125. An Example of Kernel of Non-Carleman Type

By Takashi KASUGA

(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1957)

In this note, we construct an example of symmetric measurable kernel of non-Carleman type which determines a bounded self-adjoint operator in $L^2[0, 1]$[1] and has some additional properties stated in the following.

More precisely *we construct a function* $S(x, y)$ *on* $[0, 1]\times[0, 1]$ *with the following properties* (A), (B), (C), (D), (E), (F):

(A) $S(x, y)\geqq 0,\ S(x, y)=S(y, x)$ *on* $[0, 1]\times[0, 1]$.

(B) $S(x, y)$ *is a Baire's function of the 1st class on* $[0, 1]\times[0, 1]$.

(C) *If* $f(y)\in L^2[0, 1]$, $S(x, y)f(y)\in L^1[0\leqq y\leqq 1]$[2] *for every* $x\in[0, 1]-N_f$ *where* N_f *is a null set depending on* $f(y)$.

(D) $\int_0^1 S(x, y)f(y)dy\in L^2[0, 1]$ *if* $f(y)\in L^2[0, 1]$.

(E) *The operation* H *defined for all* $f(y)\in L^2[0, 1]$ *by*

$$H: f(y)\rightarrow \int_0^1 S(x, y)f(y)dy$$

is a bounded self-adjoint operator in $L^2[0, 1]$.

But

(F) $S(x, y)\notin L^2[0\leqq y\leqq 1]$[2] *for any* $x\in[0, 1]$.

§1. Kernel $K(x, y)$. We define three functions $R(n)$, $P(n)$, $Q(n)$ of integer $n\geqq 0$ by

$$R(0)=0,\ R(n)=\sum_{s=1}^{n} s^{-1} \quad \text{for } n\geqq 1$$

$$P(n)=R(n)-[R(n)]^{3)} \quad \text{for } n\geqq 0$$

$$Q(0)=0,\ Q(n)=6\pi^{-2}\sum_{s=1}^{n} s^{-2} \quad \text{for } n\geqq 1.$$

Then since $0<R(n)-R(n-1)\leqq 1$ for $n\geqq 1$, for $n\geqq 1$ $[R(n)]=[R(n-1)]$ or $[R(n)]=[R(n-1)]+1$ and if $[R(n)]=[R(n-1)]$, then $0\leqq P(n-1)<P(n)<1$ and if $[R(n)]=[R(n-1)]+1$, then $0\leqq P(n)\leqq P(n-1)<1$. Also it is well known that $Q(n)\rightarrow 1$ $(n\rightarrow\infty)$.

We define a function $K(x, y)$ on $[0, 1]\times[0, 1]$ in the following way.

For (x, y) such that $0\leqq x\leqq 1$ $Q(n-1)\leqq y<Q(n)$ $(n\geqq 1)$, we put

---

1) $M[0, 1]$, $L[0, 1]$, $L^2[0, 1]$ are the classes of bounded measurable, integrable, square integrable functions on the closed interval $[0, 1]$ respectively.

2) $f(x, y)\in L^2[0\leqq x\leqq 1]$ or $f(x, y)\in L^2[0\leqq y\leqq 1]$ means that $f(x, y)$ as a function of x or y belongs to $L^2[0, 1]$ for a particular value of y or x. Similarly for other function classes defined in 1).

3) $[a]$ is the greatest integer not greater than the real number a.

$$K(x, y)=\begin{cases} n & \text{for } P(n-1)\leqq x\leqq P(n) \\ 0 & \text{otherwise} \end{cases}$$

if $[R(n-1)]=[R(n)]$,
and we put

$$K(x, y)=\begin{cases} n & \text{for } 0\leqq x\leqq P(n) \\ n & \text{for } P(n-1)\leqq x\leqq 1 \\ 0 & \text{otherwise} \end{cases}$$

if $[R(n-1)]+1=[R(n)]$.
We put $K(x, 1)=0$ for $x\in[0, 1]$.
That $K(x, y)$ is thus defined for all $(x, y)\in[0, 1]\times[0, 1]$, is obvious from the properties of functions $P(n)$, $Q(n)$, $R(n)$.

We can easily verify that $K(x, y)$ is a Baire's function of the 1st class on $[0, 1]\times[0, 1]$.

We take a $g(x)\in L^2[0, 1]$ and extend its domain of definition to the whole real line so that $g(x)$ becomes a function of period 1.

Then $K(x, y)g(x)\in L^1[0\leqq x\leqq 1]$ [2] for each $y\in[0, 1]$ since $K(x, y)\in M[0\leqq x\leqq 1]$ [2] for each $y\in[0, 1]$. Also considering the properties of functions $P(n)$, $Q(n)$, $R(n)$ and the definition of $K(x, y)$, we can easily verify

$$\begin{aligned}\int_0^1\left|\int_0^1 K(x, y)g(x)\,dx\right|^2 dy &= \sum_{n=1}^{\infty} 6\pi^{-2}n^{-2}\left|\int_{R(n-1)}^{R(n)} n\cdot g(x)\,dx\right|^2 \\ &= 6\pi^{-2}\sum_{n=1}^{\infty}\left|\int_{R(n-1)}^{R(n)} g(x)\,dx\right|^2 \\ &\leqq 6\pi^{-2}\sum_{n=1}^{\infty}(R(n)-R(n-1))\int_{R(n-1)}^{R(n)}|g(x)|^2 dx \\ &= 6\pi^{-1}\sum_{n=1}^{\infty} n^{-1}\int_{R(n-1)}^{R(n)}|g(x)|^2 dx. \qquad (1)\end{aligned}$$

We have

$$\begin{aligned}&6\pi^{-2}\sum_{n=1}^{M} n^{-1}\int_{R(n-1)}^{R(n)}|g(x)|^2 dx \\ &= 6\pi^{-2}\sum_{n=1}^{M} n^{-1}\left(\int_0^{R(n)}|g(x)|^2 dx-\int_0^{R(n-1)}|g(x)|^2 dx\right) \\ &= 6\pi^{-2}\left(M^{-1}\int_0^{R(M)}|g(x)|^2 dx+\sum_{n=1}^{M-1}(n^{-1}-(n+1)^{-1})\int_0^{R(n)}|g(x)|^2 dx\right) \\ &= 6\pi^{-2}\left(M^{-1}\int_0^{R(M)}|g(x)|^2 dx+\sum_{n=1}^{M-1}\{n(n+1)\}^{-1}\int_0^{R(n)}|g(x)|^2 dx\right). \qquad (2)\end{aligned}$$

If we put $W(n)=\left[\sum_{s=1}^{n} s^{-1}\right]$ [3] $=[R(n)]$ for integer $n\geqq 1$, then $W(n)=O(\log n)$ for $n\to\infty$.
Hence

$$\sum_{n=1}^{M-1}\{n(n+1)\}^{-1}\int_0^{R(n)}|g(x)|^2dx \leqq \sum_{n=1}^{M-1}\{n(n+1)\}^{-1}\{W(n)+1\}$$
$$\times\int_0^1|g(x)|^2dx=O(1) \quad \text{for } M\to\infty, \tag{3}$$

and

$$M^{-1}\int_0^{R(M)}|g(x)|^2dx \leqq M^{-1}\{W(M)+1\}\int_0^1|g(x)|^2dx=o(1)$$
$$\text{for } M\to\infty. \tag{4}$$

By (1), (2), (3) and (4), we get

$$\int_0^1\left|\int_0^1 K(x, y)g(x)dx\right|^2dy<+\infty.$$

Therefore

$$\int_0^1 K(x, y)g(x)dx \in L^2[0, 1] \tag{5}$$

for any $g(x)\in L^2[0, 1]$.

If $f(y)$, $g(x)\in L^2[0, 1]$, then by (5) and $K(x, y)\geqq 0$,

$$\int_0^1\int_0^1|K(x, y)f(y)g(x)|dxdy$$
$$=\int_0^1\left(\int_0^1 K(x, y)|g(x)|dx\right)|f(y)|dy<+\infty.$$

Hence by a theorem of Fubini and a theorem of Banach,[4)] if $f(y)\in L^2[0, 1]$, then $K(x, y)f(y)\in L^1[0\leqq y\leqq 1]$ for all $x\in[0, 1]-N_f$ where N_f is a null set depending on $f(x)$, and if we define two operators T, U by

$$T:\quad g(x)\to\int_0^1 K(x, y)g(x)dx \quad \text{for all } g(x)\in L^2[0, 1]$$

$$U:\quad f(y)\to\int_0^1 K(x, y)f(y)dy \quad \text{for all } f(y)\in L^2[0, 1],$$

then both T and U are bounded linear operators in $L^2[0, 1]$ (that is, bounded linear transformations from $L^2[0, 1]$ into $L^2[0, 1]$) and $U=T^*$ (adjoint operator of T).

§2. We shall prove in the following that $K(x, y)\notin L^2[0\leqq y\leqq 1]$ for any $x\in[0, 1]$.

We take a real number x such that $0\leqq x<1$. Then for any integer $m\geqq 0$, there is an integer $n_0\geqq 1$ such that $R(n_0-1)\leqq x+m \leqq R(n_0)$, since $R(n)\to+\infty(n\to\infty)$ and $R(n-1)<R(n)$ for any integer $n\geqq 1$. By the definitions of $P(n)$ and $R(n)$, $P(n)=R(n)-[R(n)]$ and $0<R(n)-R(n-1)\leqq 1$ for $n\geqq 1$. Hence for the above n_0, if $[R(n_0)]$

4) Cf. S. Banach [1, pp. 86-89, 104-105].

$=[R(n_0-1)]$, then $P(n_0-1)\leqq x\leqq P(n_0)$ and if $[R(n_0)]=[R(n_0-1)]+1$, then $0\leqq x\leqq P(n_0)$ or $P(n_0-1)\leqq x\leqq 1$. Also $n_0\to+\infty(m\to\infty)$. Therefore by the definition of $K(x, y)$, for each x such that $0\leqq x<1$ there are infinitely many integers $n_0\geqq 1$ such that $K(x, y)=n_0$ for $Q(n_0-1)\leqq y<Q(n_0)$.

On the other hand, there are infinitely many integers $n\geqq 1$ such that $[R(n)]=[R(n-1)]+1$ since $R(n)\to+\infty$ $(n\to\infty)$ and $0<R(n)-R(n-1)\leqq 1$ for $n\geqq 1$. Hence by the definition of $K(x, y)$, there are infinitely many integers $n_0\geqq 1$ such that $K(1, y)=n_0$ for $Q(n_0-1)\leqq y<Q(n_0)$.

Therefore for each x such that $0\leqq x\leqq 1$, there are infinitely many integers $n_0\geqq 1$ such that $K(x, y)=n_0$ for $Q(n_0-1)\leqq y<Q(n_0)$. For such n_0,

$$\int_{Q(n_0-1)}^{Q(n_0)}|K(x, y)|^2dy=n_0^2\{Q(n_0)-Q(n_0-1)\}=n_0^2\times 6\pi^{-2}\times n_0^{-2}=6\pi^{-2}.$$

Hence if $x\in[0, 1]$,

$$\int_0^1|K(x, y)|^2dy=\sum_{n=1}^{\infty}\int_{Q(n-1)}^{Q(n)}|K(x, y)|^2dy=+\infty.$$

§ 3. We put for $(x, y)\in[0, 1]\times[0, 1]$

$$S(x, y)=K(x, y)+K(y, x).$$

The function $S(x, y)$ has obviously the property (A), since $K(x, y)\geqq 0$ on $[0, 1]\times[0, 1]$. That the function $S(x, y)$ has properties (B), (C), (D), and (E) can be easily concluded from the properties of kernel $K(x, y)$ already proved. Also the function $S(x, y)$ has property (F), since $K(y, x)\in L^2[0\leqq y\leqq 1]$ for any $x\in[0, 1]$ and $K(x, y)\notin L^2[0\leqq y\leqq 1]$ for any $x\in[0, 1]$ as we have already proved.

Reference

[1] S. Banach: Théorie des opérations linéaires, Warszawa (1932).