152. A Theorem on Residuated Lattices

By Kentaro MURATA
Department of Mathematics, Yamaguchi University
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1. Let L be a complete lattice-ordered semigroup (cl-semigroup) with a maximally integral identity[1] e, and suppose that L has a unique mapping into itself $a\rightarrow a^{-1}$ with two properties 1) $aa^{-1}a\leq a$ and 2) $axa\leq a$ implies $x\leq a^{-1}$. In the previous paper [1], we obtained[2] that *L forms a commutative cl-group which is a direct product of infinite cyclic groups generated by prime elements, if L satisfies the following conditions*:

(1) *The ascending chain condition (a.c.c.) holds for integral elements of L.*

(2) *Any prime element is divisor-free (maximal).*

(3) *Any prime element contains an element c satisfying $c^{-1^{-1}}=c$.*

Our purpose of the present note is to show that the condition (1) is replaceable equivalently by *the restricted descending chain condition for integral elements of L.*

2. Let L be a cl-semigroup with an identity e. If e is maximally integral, then, in order that L has a mapping into itself $a\rightarrow a^{-1}$ with above two properties 1) and 2), it is necessary and sufficient that L forms a residuated lattice.[3] In [1] we have proved[4] that the condition is necessary. We show that the condition is sufficient. Suppose that L is a residuated lattice. Then $(e:a)_l=(e:a)_r$. For, let $ax\leq e$, then $xaxa\leq xa$, $(xa\smile e)^2\leq xa\smile e$, and so $xa\smile e=e$, $xa\leq e$. Hence $(e:a)_l\leq(e:a)_r$. Similarly $(e:a)_r\leq(e:a)_l$. We get therefore $(e:a)_l=(e:a)_r$. We next prove that $e=(a:a)_l=(a:a)_r$. Since $(a:a)_r a\leq a$, we have $(a:a)_r^2 a\leq a$, $(a:a)_r^2\leq(a:a)_r$. $(a:a)_r\geq e$ is evident. Hence $e=(a:a)_r$, similarly $e=(a:a)_l$. We now define a mapping $a\rightarrow a^{-1}$ with $a^{-1}=(e:a)_l=(e:a)_r$. Then $aa^{-1}a=a\cdot(e:a)_r a\leq ae=a$, and $axa\leq a$ implies $ax\leq(a:a)_r=e$, hence $x\leq(e:a)_l=a^{-1}$.

Lemma 1. *Let a and b be two elements in L. If b covers a, then $(a:b)_l$ is a prime element. In particular, if b is integral, then $(a:b)_l$ is a prime element containing a. Similarly for $(a:b)_r$.*

Proof. Suppose that $bx\leq a$. Then $abx\leq a^2\leq ab$. Hence $x\leq(ab:ab)_l$

1) An element x is called *integral* if $x^2\leq x$. e is called *maximally integral* if $e\leq c$ ($c^2\leq c$) implies $e=c$.

2) Cf. [1, p. 14, Theorem 2.6].

3) Cf. [2, p. 201]. $(a:b)_l$ will denote the left residual of a by b which is the largest x satisfying $bx\leq a$. Symmetrically for the right residual $(a:b)_r$ of a by b.

4) Cf. [1, p. 12, Theorem 2.2].

$=e$, i.e. $(a:b)_l$ is integral. Let u and v be two integral elements such that $uv\leq(a:b)_l$ and $u\not\leq(a:b)_l$. Then $buv\leq a$ and $bu\not\leq a$. Hence $a<bu\cup a\leq bu\cup b=b$. This implies $b=bu\cup a$, and so $bv=buv\cup av\leq a$, $v\leq(a:b)_l$. This shows that $(a:b)_l$ is prime. Similarly $(a:b)_r$ is prime. The other part of this lemma is evident.

In the following we suppose that any prime is divisor-free.

Lemma 2. *Let a be an integral element of L, and X a set of elements x such that $x^\sigma\leq a$ for a suitable whole number $\sigma=\sigma(x)$. If the descending chain condition (d.c.c.) holds for the interval $e/a=\{y; a\leq y\leq e\}$, then there exists a whole number ρ such that* $(\sup X)^\rho\leq a$.

Proof. If the set X consists of the element a only, then our assertion is trivial. We assume that X contains at least two elements. Then evidently $u=\sup X>a$. We find now that u is not an idempotent. For, let u be an idempotent. Since $eu\geq u^2=u>a$, we have $(a:u)_r\neq e$. Take an element m which covers $(a:u)_r$. Then $p=((a:u)_r:m)_l$ is a prime element, and so p is divisor-free. If $e=u\cup p$, then $e=(\sup X)\cup p=\sup_{x\in X}(x\cup p)$. Hence there exists $x_0\cup p$ $(x_0\in X)$ such that $e=x_0\cup p$. Since there exists a whole number σ such that $x_0^\sigma\leq a$, we obtain $e=e^\sigma=(x_0\cup p)^\sigma=\bigcup_{i,j}f_ipg_j\leq p$, a contradiction. Now, if $u\cup p=p$, then $u\leq p$. On the other hand, since $mpu\leq a$, we obtain $mu=mu^2\leq mpu\leq a$. Hence $m\leq(a:u)_r$. This is a contradiction. Repeating the above arguments to the set Xu,[5] we obtain $u^2=(\sup X)u=\sup(Xu)>(\sup(Xu))^2=u^4$. Continuing in this way we have $u>u^2>u^4>\cdots$, $u^\rho\leq a$.

Lemma 3. *Let a be an integral element of L. If the d.c.c. holds for the interval e/a, then a contains a product of finite number of primes.*

Proof. Let X be the set of all elements x such that $x^\sigma\leq a$ for a suitable whole number σ. Take an element c_1 which covers $u=\sup X$. Then $p_1=(u:c_1)_r\neq e$,[6] and p_1 is a prime element. If $c_1\leq p_1$, then $c_1^2\leq p_1c_1\leq u$, $c_1\in X$, a contradiction. Hence $c_1\not\leq p_1$. If $p_1\neq u$, then we take an element c_2 such that $c_2\leq p_1$ and c_2 covers u. Put $p_2=(u:c_2)_r$. Then, since $c_2\not\leq p_2$ and $c_2\leq p_1$, the prime element p_2 $(\neq e)$ is not equal to p_1. If $p_1\cap p_2\neq u$, then we take an element c_3 such that $c_3\leq p_1\cap p_2$ and c_3 covers u. Put $p_3=(u:c_3)_l$. Then p_3 $(\neq e)$ is not equal to p_1 and p_2. Continuing in this way, we obtain, after a finite number of steps, $p_1\cap\cdots\cap p_n=u$. Since there exists a whole number ρ such that $u^\rho\leq a$, we obtain

$$(p_1\cdots p_n)^\rho\leq(p_1\cap\cdots\cap p_n)^\rho=u^\rho\leq a.$$

This proves our assertion.

Lemma 4. *Suppose that the restricted descending chain condition*

5) If $x^\sigma\leq a$ $(x\in X)$, then $(xu)^\sigma\leq a$.

6) If $p_1=e$, then $c_1=ec_1=p_1c_1\leq u$, a contradiction.

(r.d.c.c.) holds for integral elements in L, and any prime contains an element c satisfying $c^{-1^{-1}}=c$. *If both a and* a^{-1} *are integral, then* $a=e$.

Proof. Let $a\leq e$, $a\neq e$. Using Lemma 1, we can take a prime element p such that $a\leq p<e$. Since $e\geq a^{-1}\geq p^{-1}\geq e^{-1}=e$, it follows that $a^{-1}=p^{-1}=e$. Let $c=c^{-1^{-1}}$ be an element contained in p, and $p_1\cdots p_\lambda$ a product of finite number of primes which is contained in c. Suppose now that λ is minimal. Then $\lambda\neq 1$. For, let $\lambda=1$, then $p_1\leq c\leq p$, $p_1=c=p$. Hence $c^{-1}=p^{-1}=e$, hence $c=e$, and hence $p=e$, a contradiction. Since $p_1\cdots p_\lambda\leq c$, there exists p_i such that $p_i\leq p$, $p_i=p$. Putting $P=p_1\cdots p_{i-1}$, $Q=p_{i+1}\cdots p_\lambda$, we have $c^{-1}P\cdot pQ\leq c^{-1}c\leq e$, and $c^{-1}P\leq (pQ)^{-1}$. On the other hand, since $pQ(pQ)^{-1}\leq e$, we have $Q(pQ)^{-1}\leq p^{-1}=e$, and $(pQ)^{-1}\leq Q^{-1}$. Hence $c^{-1}P\leq Q^{-1}$. This implies $c^{-1}PQ\leq Q^{-1}Q\leq e$, $PQ\leq c^{-1^{-1}}=c$, i.e. $p_1\cdots p_{i-1}p_{i+1}\cdots p_\lambda\leq c$, we have a contradiction to the minimality of λ.

Theorem 1. *Let L be a residuated lattice with a maximally integral identity e. Suppose that*

(1)* *The r.d.c.c. holds for integral elements of L.*

(2) *Any prime element is devisor-free.*

(3) *Any prime element contains an element c such that* $c^{-1^{-1}}=c$.

Then L forms a commutative cl-group, which is a direct product of infinite cyclic groups generated by prime elements.

Proof. $aa^{-1}\leq e$ is evident. Since $(aa^{-1})(aa^{-1})^{-1}\leq e$, we have $a^{-1}(aa^{-1})^{-1}\leq a^{-1}$, $(aa^{-1})^{-1}\leq e$. Hence $aa^{-1}=e$. L forms therefore a *cl*-group. The other part of the theorem is easily obtained. Q.E.D.

It is easy to prove the converses of Theorems 1 and 2.6 [1]. Hence we obtain the following:

Theorem 2. *Let L be a residuated lattice with a maximally integral identity. Suppose that any prime is divisor-free and contains an element c satisfying* $c^{-1^{-1}}=c$. *Then the following two conditions are equivalent.*

(1) *The a.c.c. holds for integral elements of L.*

(1)* *The r.d.c.c. holds for integral elements of L.*

By Theorem 4.5 in [1], we obtain

Theorem 3. *Let* $\mathfrak{o}$ *be a regular order in a semigroup. Suppose that* $\mathfrak{o}$ *is maximal and any closed prime* $\mathfrak{o}$*-ideal is a maximal closed two-sided* $\mathfrak{o}$*-ideal. Then the followings are equivalent:*

(A) *The a.c.c. holds for closed integral* $\mathfrak{o}$*-ideals.*

(B) *The r.d.c.c. holds for closed integral* $\mathfrak{o}$*-ideals.*

References

[1] K. Asano and K. Murata: Arithmetical ideal theory in semigroups, Jour. Institute of Polytechnics Osaka City Univ., series A, **4**, no. 1 (1953).

[2] G. Birkhoff: Lattice Theory, Amer. Math. Coll. Publ., **25** (2nd ed.) (1948).