148. On Non-linear Partial Differential Equations of Parabolic Types. III

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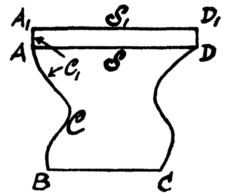
Here we give the main existence theorem and discuss on the regularity of domains. We also extend the meaning of the heat operator in the higher dimensional space in the final section.

8. Existence theorem (II). THEOREM 8.1. *Suppose that $f(x, y, u)$ is continuous, quasi-bounded with respect to u and satisfies the condition (Lk) on $(C, S]\times(-\infty, \infty)$ and the condition (P) is satisfied for the equation (E_1) and the bounded function $\beta(x, y)$ given on C. Then there exists a continuous solution of (E_1) on $(C, S]$.*

PROOF. Since $(C, S]$ is a p-domain, the end points A and D of the segment S also form end points of the curve C. Prolonging the curve C upward from A and D by length δ, we get the points A_1 and D_1. Denote the segment A_1D_1 by S_1 (not including its end points), and denote by C_1 the curve which consists of the segment A_1A, the curve C and the segment D_1D. Then $(C_1, S_1]$ is also a p-domain.



Now we extend the function $f(x, y, u)$ to $(C_1, S_1]$ as follows: if (x, y) belongs to the interior of the rectangle A_1ADD_1 or on the segment S_1 we put $f(x, y, u)=f(x, b, u)$ where b is the y-coordinate of A or D. Since there is no ambiguity, we permit ourselves to write $f(x, y, u)$ for the extended function. The function $\beta(x, y)$ given on C can be extended from C to C_1 in the same way, i.e. if (x, y) belongs to A_1A or D_1D then we put $\beta(x, y)=\beta(x, b)$. We write also $\beta(x, y)$ for the extended function.

Next we extend the Ψ_β-function $\psi(x, y)$ on $[C, S]$ to the Ψ_β-function on $[C_1, S_1]$ as follows: on the rectangle A_1ADD_1 $\psi(x, y)$ is equal to a continuous solution of (E_1) with the boundary value $\psi(x, y)$ on the closed segment AD, $\psi(A)$ on A_1A and $\psi(D)$ on D_1D. This continuous solution does exist. Indeed, to find such a solution we shall first solve the equation of heat conduction with the given boundary condition, and let $h(x, y)$ be a solution of it. We shall consider the equation $\square v=f(x, y, v+h(x, y))$. Since $f(x, y, u)$ is quasi-bounded with respect to u on $(C, S]$, $f(x, y, v+h(x, y))$ is also quasi-bounded with respece to v on $(C_1, S_1]$, therefore by Theorem 4.2 there is a solution $v(x, y)$ satisfying $\square v=f(x, y, v+h(x, y))$ and vanishing on the boundary.

Then, $v(x, y)+h(x, y)$ is the desired solution of (E_1). The function $\psi(x, y)$ extended in this way is evidently a Ψ_β-function on $[C_1, S_1]$. By an analogous process we can extend Φ_β-function on $[C, S]$ to Φ_β-function on $[C_1, S_1]$. Thus the condition (P) is satisfied for the equation (E_1) on $[C_1, S_1]$.

Now for any $(x, y)\in(C_1, S_1]$, set

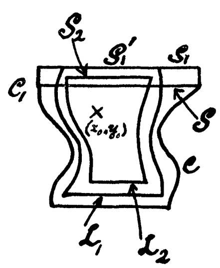
$$u(x, y)=\inf\{\psi(x, y);\ \psi\in\Psi_\beta \quad \text{on } [C_1, S_1]\}.$$

Let $\{(x_n, y_n);\ n=1, 2, 3, \cdots\}$ be a countable dense set of points in $(C_1, S_1]$. Then there exists a sequence $\{\psi_n(x, y)\}$ of Ψ_β-functions on $[C_1, S_1]$ such that

i) $\psi_1(x, y)\geq\psi_2(x, y)\geq\psi_3(x, y)\geq\cdots$

ii) $\lim\limits_{n\to\infty}\psi_n(x_i, y_i)=u(x_i, y_i), \quad i=1, 2, 3, \cdots$.

Let (x_0, y_0) be any point in $(C, S]$. Then there exist $(\mathcal{L}_1, S_1']$ and $(\mathcal{L}_2, S_2]$ such that $S_1'\subset S_1$, $(x_0, y_0)\in(\mathcal{L}_2, S_2)$, $[\mathcal{L}_2, S_2]\subset(\mathcal{L}_1, S_1']$ and the y-coordinate of S_2 is greater than that of S and less than that of S_1. We have $M_{\mathcal{L}_1}\psi_1(x, y)\geq M_{\mathcal{L}_1}\psi_2(x, y)\geq\cdots$, and $\psi_n(x, y)\geq M_{\mathcal{L}_1}\psi_n(x, y)\geq u(x, y)$ on $(C_1, S_1]$. By Theorem 7.4 we have $M_{\mathcal{L}_1}\psi_n\in\Psi_\beta$, and by Theorem 7.1 $M_{\mathcal{L}_1}\psi_n(x, y)\geq\varphi(x, y)$, where $\varphi\in\Phi_\beta$, so that $\{M_{\mathcal{L}_1}\psi_n(x, y)\}$ is bounded. Since $M_{\mathcal{L}_1}\psi_n$ is a solution of (E_1) in $(\mathcal{L}_1, S_1']$, by Theorem 5.3,[1] $\{M_{\mathcal{L}_1}\psi_n\}$ converges uniformly to a continuous solution $U(x, y)$ of (E_1) on $[\mathcal{L}_2, S_2]$.



For the points (x_i, y_i) contained in $(\mathcal{L}_2, S_2]$ we have $u(x_i, y_i)=U(x_i, y_i)$. By applying the same method to the set of points $\{(x_i, y_i);\ i=0, 1, 2, \cdots\}$, if we have $U_1(x, y)$ like $U(x, y)$ above, then

$$U_1(x_0, y_0)=u(x_0, y_0), \qquad U_1(x_i, y_i)=u(x_i, y_i)$$

hold true for the points (x_i, y_i) contained in $(\mathcal{L}_2, S_2]$. Since $U(x, y)$ and $U_1(x, y)$ are continuous on $(\mathcal{L}_2, S_2]$, we have $U(x_0, y_0)=U_1(x_0, y_0)=u(x_0, y_0)$. Since (x_0, y_0) is an arbitrary point in $(C, S]$, $u(x, y)$ is a continuous solution of (E_1) on $(C, S]$. Q.E.D.

9. Barriers. DEFINITION. *Suppose that $f(x, y, u)$ is quasi-bounded with respect to u on $(C, S]\times(-\infty, \infty)$. A continuous function $w(x, y)$ on $[C, S]$ is said to be a barrier of (E_1) with respect to $\beta(x, y)$ at the point $(x_0, y_0)\in C$ if*

i) $w(x, y)>0 \qquad (x, y)\in[C, S], \quad (x, y)\neq(x_0, y_0)$,

ii) $w(x, y)\to0$ *as* $(x, y)\to(x_0, y_0), \quad (x, y)\in[C, S]$,

1) Since we have $\varphi(x, y)\leq M_{\mathcal{L}_1}\varphi_n(x, y)\leq\varphi_1(x, y)$, by setting

$$g(x, y, u)=\begin{cases} f(x, y, \varphi_1(x, y)) & \varphi_1(x, y)<u \\ f(x, y, u) & \varphi(x, y)\leq u\leq\varphi_1(x, y) \\ f(x, y, \varphi(x, y)) & u<\varphi(y, x), \end{cases}$$

it suffices to consider the equation $\Box u=g(x, y, u)$ instead of (E_1). Here $g(x, y, u)$ is bounded, so that we can apply Theorem 5.3.

iii) $\overline{\Box} w(x, y) \leq -M$, *where*

$$M = \sup_{(x, y) \in (C, S]} \{ | f(x, y, \bar{\beta}(x_0, y_0)) |, \ | f(x, y, \underline{\beta}(x_0, y_0)) | \}.$$

THEOREM 9.1. *Suppose that $f(x, y, u)$ is continuous, quasi-bounded with respect to u and satisfies the condition (Lk) on $(C, S] \times (-\infty, \infty)$. If there exists a barrier of (E_1) with respect to $\beta(x, y)$ at $(x_0, y_0) \in C$, then (E_1) satisfies the condition (P) and we have*

$$\underline{\beta}(x_0, y_0) \leq \underline{u}(x_0, y_0) \leq \bar{u}(x_0, y_0) \leq \bar{\beta}(x_0, y_0) \tag{9.1}$$

where $u(x, y)$ is the global solution in Theorem 8.1.

PROOF. Let ε be an arbitrary positive number. Setting

$$\psi(x, y) = \bar{\beta}(x_0, y_0) + \varepsilon + K_1 w(x, y),$$

$\psi(x, y)$ is a Ψ_β-function of (E_1) on $[C, S]$ if K_1 is sufficiently large. Indeed, for the sufficiently small neighbourhood U of (x_0, y_0) $\psi(x, y) \geq \bar{\beta}(x_0, y_0) + \varepsilon \geq \bar{\beta}(x, y)$ on $C \frown U$, and since on $C - U$ there exists $\alpha > 0$ such that $w(x, y) \geq \alpha > 0$, by the boundedness of $\beta(x, y)$, we have $\psi(x, y) \geq \bar{\beta}(x, y)$ for sufficiently large K_1. Therefore $\psi(x, y) \geq \bar{\beta}(x, y)$ on C. Next it follows from the inequality $w(x, y) \geq 0$ that $\psi(x, y) \geq \bar{\beta}(x_0, y_0)$ on $(C, S]$. Hence

$$f(x, y, \psi(x, y)) \geq f(x, y, \bar{\beta}(x_0, y_0)) \geq -M.$$

Since $\overline{\Box} \psi(x, y) = K_1 \overline{\Box} w(x, y) \leq -K_1 M$, if $K_1 \geq 1$, we have $\overline{\Box} \psi(x, y) \leq -M \leq f(x, y, \psi(x, y))$ on $(C, S]$. Thus we have proved that $\psi(x, y)$ is a Ψ_β-function on $[C, S]$ if K_1 is sufficiently large. Similarly we can prove that

$$\varphi(x, y) = \underline{\beta}(x_0, y_0) - \varepsilon - K_2 w(x, y)$$

is a Φ_β-function on $[C, S]$ if K_2 is sufficiently large. Thus the condition (P) is satisfied.

Since $w(x_0, y_0) = 0$, there exist small neighbourhoods U_1 and U_2 of (x_0, y_0) such that

$$\psi(x, y) \leq \bar{\beta}(x_0, y_0) + 2\varepsilon \quad \text{if } (x, y) \in [C, S] \frown U_1$$
$$\varphi(x, y) \geq \underline{\beta}(x_0, y_0) - 2\varepsilon \quad \text{if } (x, y) \in [C, S] \frown U_2.$$

In Theorem 8.1 we set $u(x, y) = \inf \{\psi(x, y);\ \psi \in \Psi_\beta \text{ on } [C_1, S_1]\}$, where $[C_1, S_1]$ was an extended p-domain. However, the method of extending the Ψ_β-functions from those on $[C, S]$ to those on $[C_1, S_1]$ shows that on $[C, S]$ we have $u(x, y) = \inf \{\psi(x, y);\ \psi \in \Psi_\beta \text{ on } [C, S]\}$. Therefore, we have

$$\underline{\beta}(x_0, y_0) - 2\varepsilon \leq u(x, y) \leq \bar{\beta}(x_0, y_0) + 2\varepsilon \quad \text{on } [C, S] \frown U_1 \frown U_2.$$

Hence we have (9.1). Q.E.D.

REMARK. Since $\beta(x, y) \leq \psi(x, y)$ on C, we have $\bar{\beta}(x_0, y_0) = \bar{u}(x_0, y_0)$ in (9.1).

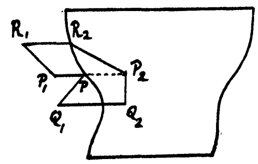
COROLLARY. *If $\beta(x, y)$ is continuous at (x_0, y_0), then $u(x_0, y_0) = \beta(x_0, y_0)$.*

EXAMPLE. *We shall give here an example of a p-domain* $[C, \mathcal{S}]$ *which has barriers at each point of* C.

Suppose that C *consists of two side curves expressed by* $x=\lambda_1(y)$ *and* $x=\lambda_2(y)$, $a\leq y\leq b$, *and a lower bounding segment. If* λ_1 *and* λ_2 *satisfy Lipschitz's condition locally from the left on* $a<y\leq b$, *there exists a barrier at each point of* C. *In other words, if we can drive at each point of* C *in a wedge whose upper bounding segment is parallel to the x-axis, we can construct a barrier at each point of* C.

We shall prove it.

Case 1. $P(x_0, y_0)$ *is on the side boundary curves except their end points.* Since $\lambda_i(y)$ satisfies Lipschitz's condition locally from the left, for sufficiently small $\varepsilon>0$, on the line $y=y_0-\varepsilon$ we can take two points Q_1 and Q_2 such that the segment PQ_1 lies in the exterior of $[C, \mathcal{S}]$ except the point P and such that Q_2 is in the interior of $(C, \mathcal{S}]$. Next on the line $y=y_0$ we can take an exterior point P_1 and an interior point P_2 such that the segment P_2Q_2 is wholly in $(C, \mathcal{S}]$. Finally on the line $y=y_0+\varepsilon$ we can take two points R_1 and R_2 such that R_1 is in the exterior and R_2 is on the side curve and moreover the segment R_2P_2 is wholly in the interior of $(C, \mathcal{S}]$ except the point R_2. We denote the broken line $R_1P_1PQ_1Q_2P_2R_2$ by $\mathcal{B}$ and the open segment R_1R_2 by $\mathcal{S}_1$.



Let M_1 be a constant greater than M where M is the constant in the definition of barrier. If we take Γ so large that $x^2-2y-\Gamma$ is negative on $[C, \mathcal{S}]\smile[\mathcal{B}, \mathcal{S}_1]$, then

$$\omega(x, y)=-M_1\left(\frac{x^2-2y-\Gamma}{4}\right)$$

is positive on the domain $(C, \mathcal{S}]\smile[\mathcal{B}, \mathcal{S}_1]$. Now we give as a boundary value the following continuous function on $\mathcal{B}$: on R_1P_1 it is equal to $\omega(x, y)$, on P_1P it is equal to a continuous function which, not being greater than $\omega(x, y)$, equal to $\omega(P_1)$ at P_1, equal to 0 at P and positive except P, varies continuously from $\omega(P_1)$ to 0 as (x, y) varies from P_1 to P, and on the segment PQ_1 it is equal to a continuous function which has similar properties and varies from $\omega(Q_1)$ to 0 as (x, y) varies from Q_1 to P, and finally it is equal to $\omega(x, y)$ on the broken line $Q_1Q_2P_2R_2$. Let $v(x, y)$ be a solution of $\Box v=-M$ which is continuous on $[\mathcal{B}, \mathcal{S}_1]$ and which admits the boundary value above, then $v(x, y)$ is positive except P and not greater than $\omega(x, y)$ on $[\mathcal{B}, \mathcal{S}_1]$. If we define $w(x, y)$ as $v(x, y)$ on $[\mathcal{B}, \mathcal{S}_1]$ and as $\omega(x, y)$ on $[C, \mathcal{S}]-[\mathcal{B}, \mathcal{S}_1]$, it is easily seen that $w(x, y)$ is a barrier at $P(x_0, y_0)$.

Case 2. $P(x_0, y_0)$ *is at the end point of the side curve.* If P is at the upper end, it is sufficient to construct $v(x, y)$ on the trapezoid $PQ_1Q_2P_2$. If P is at the lower end, it suffices to construct $v(x, y)$ on the trapezoid $R_1P_1PP_2R_2$.

The detail discussion is similar to Case 1.

Case 3. *$P(x_0, y_0)$ is on the lower bounding segment.* In such a case, set $v(x, y)=(x-x_0)^2+(2+M)(y-y_0)$. Then, $\Box v(x, y)=2-(2+M)=-M$, $v(x, y)>0$ on $[C, S]$ if $(x, y)\neq(x_0, y_0)$ and $v(x_0, y_0)=0$.

10. Regularity of domains. DEFINITION. *We say that $(x_0, y_0)\in C$ is a regular point with respect to (E_1) and $\beta(x, y)$ if there is a barrier of (E_1) with respect to $\beta(x, y)$ at the point (x_0, y_0).*

From the definition of the barrier in the previous section, we see directly that the barrier of (E_1) is also a barrier of the equation $\Box u=0$, i.e. the barrier for $M=0$ in our sense. This barrier is just what B. Pini defined in his paper.[2] Therefore, we see by his theorem that if C consists of regular points in our sense, there exists a solution of the equation of heat conduction which admits the prescribed continuous boundary value. We shall prove the converse of this.

THEOREM 10.1. *Let $f(x, y, u)$ be quasi-bounded with respect to u on $(C, S]\times(-\infty, \infty)$. If at each point of C there is a barrier of the equation of heat conduction, then every point of C is the regular point of (E_1) with respect to $\beta(x, y)$ where $\beta(x, y)$ is an arbitrary bounded function on C.*

PROOF. Set $M=\sup\limits_{(x, y)\in(C, S]}\{|f(x, y, \bar{\beta}(x_0, y_0))|, |f(x, y, \underline{\beta}(x_0, y_0))|\}$, where (x_0, y_0) is an arbitrary point on C. By the assumption we have a barrier with respect to the equation of heat conduction, therefore there exists a function $v(x, y)$ such that

i) $v(x, y)$ is continuous on $[C, S]$,

ii) $v(x, y)>0$ on $[C, S]$ except at (x_0, y_0),

iii) $v(x, y)\to 0$ as $(x, y)\to(x_0, y_0)$, $(x, y)\in[C, S]$,

iv) $\overline{\Box} v(x, y)\leq 0$.

Now, since C consists of regular points of the equation of heat conduction, there exists a solution $v_1(x, y)$ of $\Box v=-M$ which is continuous on $[C, S]$ and which vanishes on C. The solution $v_1(x, y)$ is non-negative. Setting

$$w(x, y)=v(x, y)+v_1(x, y),$$

$w(x, y)$ is barrier of (E_1). Q.E.D.

11. The extension to higher dimensional spaces

For the operator $\sum\limits_{i=1}^{n}\dfrac{\partial^2}{\partial x_i^2}-\dfrac{\partial}{\partial t}$, we shall define *a generalized heat operator* $\Box$ in the $(n+1)$-dimensional space as follows. Let $S_{P,r}$ be a surface defined by the expressions

2) B. Pini: Sulla soluzione generalizzata di Wiener per il primo problema nel caso parabolico, Rend. Sem. Mat. Padova (1954).

$$\begin{aligned}
\xi_1 &= x_1 + 2r\sqrt{n}\sin\theta\sqrt{\log\operatorname{cosec}\theta}\,\eta_1 \\
\xi_2 &= x_2 + 2r\sqrt{n}\sin\theta\sqrt{\log\operatorname{cosec}\theta}\,\eta_2 \\
&\cdots\cdots\cdots\cdots \\
\xi_n &= x_n + 2r\sqrt{n}\sin\theta\sqrt{\log\operatorname{cosec}\theta}\,\eta_n \\
\tau &= t - r^2\sin^2\theta \qquad \left(0 \le \theta \le \frac{\pi}{2}\right)
\end{aligned}$$

where $P=P(x_1, x_2, \cdots, x_n, t)$, $Q=(\xi_1, \xi_2, \cdots, \xi_n, \tau)$ and

$$\begin{aligned}
\eta_1 &= \cos\varphi_1\cos\varphi_2\cdots\cos\varphi_{n-2}\cos\varphi_{n-1} \\
\eta_2 &= \cos\varphi_1\cos\varphi_2\cdots\cos\varphi_{n-2}\sin\varphi_{n-1} \\
\eta_3 &= \cos\varphi_1\cos\varphi_2\cdots\sin\varphi_{n-2} \\
&\cdots\cdots \\
\eta_{n-1} &= \cos\varphi_1\sin\varphi_2 \\
\eta_n &= \sin\varphi_1,
\end{aligned}$$

where $-\frac{\pi}{2} \leqq \varphi_i \leqq \frac{\pi}{2}$ $(i=1, 2, \cdots, n-2)$ and $0 \leqq \varphi_{n-1} \leqq 2\pi$.

We define $\overline{\square} u$ and $\underline{\square} u$ by the expressions

$$\overline{\square} u = \overline{\lim_{r\downarrow 0}}\, \frac{(n+2)^{\frac{n}{2}+1}}{n\pi^{\frac{n}{2}} r^2} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{u(Q)-u(P)\} \sin^{n-1}\theta\cos\theta$$
$$\times (\log\operatorname{cosec}\theta)^{\frac{n}{2}} J\, d\varphi_1 \cdots d\varphi_{n-1} d\theta,$$

$$\underline{\square} u = \underline{\lim_{r\downarrow 0}}\, \frac{(n+2)^{\frac{n}{2}+1}}{n\pi^{\frac{n}{2}} r^2} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \{u(Q)-u(P)\} \sin^{n-1}\theta\cos\theta$$
$$\times (\log\operatorname{cosec}\theta)^{\frac{n}{2}} J\, d\varphi_1 \cdots d\varphi_{n-1} d\theta,$$

where
$$J = \det \begin{vmatrix} \eta_1 & \eta_2 & \cdots\cdots & \eta_n \\ \dfrac{\partial\eta_1}{\partial\varphi_1} & \dfrac{\partial\eta_2}{\partial\varphi_1} & \cdots\cdots & \dfrac{\partial\eta_n}{\partial\varphi_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\eta_1}{\partial\varphi_{n-1}} & \dfrac{\partial\eta_2}{\partial\varphi_{n-1}} & \cdots\cdots & \dfrac{\partial\eta_n}{\partial\varphi_{n-1}} \end{vmatrix}$$

If $\overline{\square} u$ and $\underline{\square} u$ coincide, we denote it by $\square u$. This generalized heat operator $\square$ has the same properties listed in Section 1. All results for (E_1), (E_2) and (E_3) given in Sections 2 to 10, except the Example in Section 9, hold true for the equations

$$\square u = f(x_1, \cdots, x_n, t, u, \partial_{x_1} u, \cdots, \partial_{x_n} u, \partial_t u),$$
$$\square u = f(x_1, \cdots, x_n, t, u, \partial_{x_1} u, \cdots, \partial_{x_n} u)$$

and

$$\square u = f(x_1, \cdots, x_n, t, u)$$

respectively.