147. On Non-linear Partial Differential Equations of Parabolic Types. II

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As stated in the Introduction of the previous paper,[1] we give here some uniqueness conditions, existence theorems (I) and some preparatory theorems for the main existence theorem which will be given in the next part.

3. Uniqueness conditions. LEMMA. *Let $f(x,y,u,p)$ be defined on $(x,y)\in(C,S]$, $-\infty<u,p<+\infty$. If*

$$f(x,y,u,p)\begin{cases}>0 & u>0\\ =0 & u=0\\ <0 & u<0,\end{cases}\tag{3.1}$$

then there is one and only one solution of (E_2) which is continuous on $[C,S]$ and which vanishes on C.

DEFINITION. *Let $f(x,y,u,p)$ be a function defined on $(x,y)\in(C,S]$, $-\infty<u,\ p<+\infty$. We say that $f(x,y,u,p)$ satisfies the condition (Lk) if there exists a positive constant k such that*

$$f(x,y,u_1,p)-f(x,y,u_2,p)>-k(u_1-u_2)\tag{Lk}$$

for $(x,y)\in(C,S]$ and $u_1>u_2$.

REMARK. If we set $v=ue^{-ky}$, by simple calculation, we have

$$\begin{aligned}\overline{\square}v(x,y)&\le ke^{-ky}u(x,y)+e^{-ky}\overline{\square}u(x,y),\\ \underline{\square}v(x,y)&\ge ke^{-ky}u(x,y)+e^{-ky}\underline{\square}u(x,y).\end{aligned}\tag{3.2}$$

Then the equation (E_2) is written by

$$\square v=F(x,y,v,\partial_x v)\tag{3.3}$$

where

$$F(x,y,v,p)=kv+e^{-ky}f(x,y,ve^{ky},pe^{ky}).\tag{3.4}$$

If we assume that $f(x,y,u,p)$ satisfies the condition (Lk)

$$\begin{aligned}&F(x,y,v_1,p)-F(x,y,v_2,p)\\ &=k(v_1-v_2)-e^{-ky}\{f(x,y,v_1e^{ky},pe^{ky})-f(x,y,v_2e^{ky},pe^{ky})\}\\ &>k(v_1-v_2)-k(v_1-v_2)=0\end{aligned}$$

for $v_1>v_2$, so that $F(x,y,v,p)$ is *monotone increasing* (*strictly*) *with respect to v.*

B. Pini proved in his paper[2] that (E_2) has at most one solution which is continuous on $[C,S]$ and which admits the prescribed continuous boundary value if $f(x,y,u,p)$ is monotone increasing with

1) Proc. Japan Acad., **33**, 530–535 (1957).

2) B. Pini: Sul primo problema di valori al contorno per l'equazione parabolica non lineare del secondo ordine, Rend. del Sem. Mat. Università di Padova, 153 (1957).

respect to u. Therefore we have

THEOREM 3.1. *If $f(x,y,u,p)$ satisfies the condition (Lk) there is at most one solution of (E_2) which is continuous on $[\mathcal{C},\mathcal{S}]$ and which admits the prescribed continuous boundary value on $\mathcal{C}$.*

4. Existence theorems (I). THEOREM 4.1.[3] *Let $(\mathcal{L},\mathcal{S}]$ be a C^1-p-domain. If $f(x,y,u,p)$ is bounded and continuous on $(x,y)\in(\mathcal{L},\mathcal{S}]$, $-\infty<u,\ p<+\infty$, then (E_2) has at least one solution which is continuous on $[\mathcal{L},\mathcal{S}]$ and which vanishes on $\mathcal{L}$.*

THEOREM 4.2. *Suppose that $f(x,y,u,p)$ is quasi-bounded with respect to u on $(x,y)\in(\mathcal{L},\mathcal{S}]$, $-\infty<u,\ p<+\infty$, and moreover $f(x,y,u,p)$ satisfies the condition (Lk) there. Then, (E_2) has at least one solution which is continuous on $[\mathcal{L},\mathcal{S}]$ and which vanishes on $\mathcal{L}$.*

PROOF. As we mentioned in the last section, under the condition (Lk) we can assume without loss of generality that $f(x,y,u,p)$ is monotone increasing with respect to u. Since

$$f(x,y,u,p)=f(x,y,u,p)-f(x,y,0,p)+f(x,y,0,p),$$

from Corollary 1 of Theorem 2.7, there is a constant $M>0$ such that every solution $u(x,y)$ of (E_2) which vanishes on $\mathcal{C}$ and which is continuous on $[\mathcal{C},\mathcal{S}]$ satisfies $|u(x,y)|\leq M$ if they exist. Set

$$g(x,y,u,p)=\begin{cases} f(x,y,M,p) & u>M \\ f(x,y,u,p) & M\geq u\geq -M \\ f(x,y,-M,p) & -M>u. \end{cases} \tag{4.1}$$

Then solutions of (E_2) are solutions of

$$\Box u=g(x,y,u,\partial_x u) \tag{4.2}$$

and *vise versa*. Since $g(x,y,u,p)$ is bounded, Theorem 4.1 shows that there is at least one solution. Q.E.D.

5. Harnack's theorems. In the sequel, we assume that $f(x,y,u,p)$ satisfies the condition (Lk). So we can assume without loss of generality that $f(x,y,u,p)$ is monotone increasing with respect to u.

THEOREM 5.1. *Let $\{u_n(x,y)\}$ be a sequence of solutions of (E_2) which are continuous on $[\mathcal{C},\mathcal{S}]$. If $\{u_n(x,y)\}$ converges uniformly on $\mathcal{C}$, then it converges also uniformly on $[\mathcal{C},\mathcal{S}]$.*

PROOF. Set $u_{n,p}(x,y)=u_{n+p}(x,y)-u_n(x,y)$. Then $u_{n,p}(x,y)$ satisfies $\Box u=g(x,y,u,\partial_x u)$, where $g(x,y,u,p)=f(x,y,u+u_n(x,y),\ \partial_x u+\partial_x u_n(x,y))-f(x,y,u_n(x,y),\ \partial_x u_n(x,y))$, so that

$$g(x,y,0,0)\begin{cases} >0 & u>0 \\ =0 & u=0 \\ <0 & u<0. \end{cases}$$

Since $\{u_n(x,y)\}$ converges uniformly on $\mathcal{C}$, for any $\varepsilon>0$ there exists N such that

$$|u_{n,p}(x,y)|=|u_{n+p}(x,y)-u_n(x,y)|<\varepsilon$$

3) This theorem is an immediate consequence of Theorem 6, B. Pini (loc. cit. p. 158), so we omit the proof here.

on C for $n>N$. By Theorem 2.1[bis] we see that the inequality also holds on $(C, S]$. This shows the uniform convergence of $\{u_n(x, y)\}$ on $[C, S]$. Q.E.D.

THEOREM 5.2. *Suppose that $f(x, y, u, p)$ is quasi-bounded with respect to u on $(x, y) \in (\mathcal{L}, S]$, $-\infty<u, p<+\infty$. Let $\{u_n(x, y)\}$ be a sequence of solutions of (E_2) which are continuous on $[\mathcal{L}, S]$. If $\{u_n(x, y)\}$ converges uniformly on $\mathcal{L}$, then it converges also uniformly on $[\mathcal{L}, S]$ and the limit function $u(x, y)$ is a solution of (E_2) on $(\mathcal{L}, S]$.*

PROOF. By the previous theorem, $\{u_n(x, y)\}$ converges uniformly to a continuous function $u(x, y)$ on $[\mathcal{L}, S]$. Let $h_n(x, y)$ be the solution of $\Box u=0$ which is continuous on $[\mathcal{L}, S]$ and which admits the boundary value $u_n(x, y)$ on $\mathcal{L}$, and let $h(x, y)$ be the solution of $\Box u=0$ which is continuous on $[\mathcal{L}, S]$ and admits the boundary value $\{u(x, y)\}$ on $\mathcal{L}$. Then $\{h_n(x, y)\}$ converges to $h(x, y)$ uniformly on $[\mathcal{L}, S]$. Now, set $v_n(x, y)=u_n(x, y)-h_n(x, y)$, $v(x, y)=u(x, y)-h(x, y)$. Then $\{v_n(x, y)\}$ converges uniformly to $v(x,y)$ on $[\mathcal{L}, S]$. Since $u_n(x,y)$ are equi-bounded on $\mathcal{L}$, by the same way as in (4.1), we see that $u_n(x,y)$ are equi-bounded on $[\mathcal{L}, S]$. Hence $h_n(x, y)$ are also equi-bounded on $[\mathcal{L}, S]$. Therefore $v_n(x, y)$ are equi-bounded on $[\mathcal{L}, S]$. So that, in the expressions

$$(5.1) \qquad v_n(x, y)=\iint_{[\mathcal{L}, S]} G(x, y; \xi, \eta) f(\xi, \eta, v_n(\xi, \eta)+h_n(\xi, \eta), \partial_x v_n(\xi, \eta) + \partial_x h_n(\xi, \eta)) d\xi\, d\eta,$$

we can assume that $f(\xi, \eta, v_n(\xi, \eta)+h_n(\xi, \eta), \partial_x v_n(\xi, \eta)+\partial_x h_n(\xi, \eta))$ are equi-bounded and we can prove easily from this fact that $\{v_n(x, y)\}$ and $\{\partial_x v_n(x, y)\}$ are equi-continuous. Therefore $v(x, y)$ is differentiable with respect to x and $\{\partial_x v_n(x, y)\}$ converges uniformly to $\partial_x v(x,y)$. From (5.1) it follows

$$v(x, y)=\iint_{[\mathcal{L}, S]} G(x, y; \xi, \eta) f(\xi, \eta, v(\xi, \eta)+h(\xi, \eta), \partial_x v(\xi, \eta)+\partial_x h(\xi, \eta)) d\xi\, d\eta.$$

This expression shows that $v(x, y)$ is a solution of

$$\Box v=f(x, y, v+h(x, y), \partial_x v+\partial_x h(x, y))$$

and $v(x, y)$ vanishes on $\mathcal{L}$. Therefore $u(x, y)$ is a solution of (E_2). Q.E.D.

THEOREM 5.3. *Suppose that $f(x, y, u, p)$ is bounded, continuous and satisfies the condition (Lk) on $(x, y) \in (C, S]$, $-\infty<u, p<+\infty$. Let $\{u_n(x, y)\}$ be a non-decreasing sequence of solutions of (E_2) on $(C, S]$. Moreover suppose that there exists a point (x_0, y_0) in $(C, S]$ such that $\{u_n(x_0, y_0)\}$ is bounded. Then, $\{u_n(x, y)\}$ converges uniformly in the wider sense in $(C, S)_{y_0}$, and the limit function $u(x, y)$ is a solution of (E_2) in $(C, S)_{y_0}$. Moreover, $\{\partial_x u_n(x, y)\}$ converges uniformly to $\partial_x u(x, y)$ in $(C, S)_{y_0}$.*

PROOF. To prove the uniform convergence in the wider sense in

$(\mathcal{C}, \mathcal{S})_{y_0}$, we must prove the uniform convergence on any compact set $K \subset (\mathcal{C}, \mathcal{S})_{y_0}$, but in this case for such K we can take a C^1-p-domain $[\mathcal{L}, \mathcal{S}']$ such that $K \subset [\mathcal{L}, \mathcal{S}'] \subset (\mathcal{C}, \mathcal{S})_{y_0}$, so that it suffices to prove the uniform convergence of $\{u_n(x, y)\}$ on $[\mathcal{L}, \mathcal{S}']$. Similar discussion shows that it is sufficient to prove that the limit function is a continuous solution of (E_2) in $(\mathcal{L}, \mathcal{S}']$.

Now take a C^1-p-domain $(\mathcal{L}', \mathcal{S}'']$ such that $\mathcal{S}'' \subset \mathcal{S}$, $[\mathcal{L}, \mathcal{S}'] \subset (\mathcal{L}', \mathcal{S}''] \subset (\mathcal{C}, \mathcal{S}]$ and $(x_0, y_0) \in (\mathcal{L}', \mathcal{S}'']$. Let $h_n(x, y)$ be solutions of $\square h = 0$ which are continuous on $[\mathcal{L}', \mathcal{S}'']$ and which admit their boundary values $u_n(x, y)$ on $\mathcal{L}'$, then $h_n(x, y)$ increase with n on $[\mathcal{L}', \mathcal{S}'']$ since they are so on $\mathcal{L}'$. Setting $v_n(x, y) = u_n(x, y) - h_n(x, y)$, we see that $v_n(x, y)$ are solutions of

$$\square v = f(x, y, v + h_n(x, y), \partial_x v + \partial_x h_n(x, y))$$

and $v_n(x, y)$ vanish on $\mathcal{L}'$. Since the right hand of this expression is bounded, by Corollary 1 or 2 of Theorem 2.7 $v_n(x, y)$ are bounded, so that $h_n(x_0, y_0) = u_n(x_0, y_0) - v_n(x_0, y_0)$ are also bounded. By Harnack's second theorem for the equation of heat conduction,[4] $\{h_n(x, y)\}$ converges uniformly to a solution $h(x, y)$ of $\square h = 0$ on $[\mathcal{L}, \mathcal{S}']$. Since $u_n(x, y) = v_n(x, y) + h_n(x, y)$, $u_n(x, y)$ are bounded, so that $\{u_n(x, y)\}$ converges to a limit function $u(x, y)$.

We can prove the equi-continuity of $v_n(x, y)$ and $\partial_x v_n(x, y)$ in the same way as in the proof of the previous theorem, so that $\{v_n(x, y)\}$ converges uniformly to $v(x, y)$ and $\{\partial_x v_n(x, y)\}$ converges uniformly to $\partial_x v(x, y)$ in $[\mathcal{L}, \mathcal{S}']$. Now, from the expression

$$v_n(x, y) = \iint_{[\mathcal{L}, \mathcal{S}']} G(x, y; \xi, \eta) f(\xi, \eta, v_n(\xi, \eta) + h_n(\xi, \eta), \partial_x v_n(\xi, \eta) + \partial_x h_n(\xi, \eta)) d\xi\, d\eta$$

it follows by letting $n \to \infty$ that

$$v(x, y) = \iint_{[\mathcal{L}, \mathcal{S}']} G(x, y; \xi, \eta) f(\xi, \eta, v(\xi, \eta) + h(\xi, \eta), \partial_x v(\xi, \eta) + \partial_x h(\xi, \eta)) d\xi\, d\eta.$$

Therefore $u(x, y) = v(x, y) + h(x, y)$ is a solution of (E_2). Q.E.D.

6. Quasi-superior and quasi-inferior functions. In the sequal unless we give special attention we assume that $f(x, y, u)$ is defined over $(\mathcal{C}, \mathcal{S}] \times (-\infty, \infty)$ or $(\mathcal{L}, \mathcal{S}] \times (-\infty, \infty)$ and satisfies the condition (Lk) with respect to u. So that we can assume without loss of generality that $f(x, y, u)$ is increasing (strictly) with respect to u.

DEFINITION. *We say that $\omega(x, y)$ is a majorant function of (E_1) on $[\mathcal{C}, \mathcal{S}]$, if*

i) *$\omega(x, y)$ is continuous on $[\mathcal{C}, \mathcal{S}]$,*

ii) *if $\omega(x, y) \geq u(x, y)$ on $\mathcal{C}$, this inequality holds also on $(\mathcal{C}, \mathcal{S}]$, where*

4) B. Pini: Sulla soluzione generalizzata di Wiener per il primo problema nel caso parabolico, Rend. Sem. Mat. Padova (1954).

$u(x, y)$ is a solution of (E_1) which is continuous on $[C, S]$.

DEFINITION. *We say that $\Omega(x, y)$ is quasi-superior with respect to (E_1) at a point $(x, y) \in (C, S]$, if*

i) *$\Omega(x, y)$ is continuous at (x, y),*

ii) $\overline{\Box}\Omega(x, y) \leq f(x, y, \Omega(x, y))$.

We say that $\Omega(x, y)$ is a quasi-superior function of (E_1) on $[C, S]$, if

i) *$\Omega(x, y)$ is continuous on $[C, S]$,*

ii) *$\Omega(x, y)$ is quasi-superior with respect to (E_1) at any point of $(C, S]$.*

Minorant functions and quasi-inferior functions are defined analogously.

From the comparison theorems in Section 2, we have

THEOREM 6.1. *Quasi-superior function is majorant function.*

THEOREM 6.2. *If $\Omega_1, \Omega_2, \cdots, \Omega_n$ are quasi-superior functions of (E_1) on $[C, S]$, then $\Omega = \mathrm{Min}\{\Omega_1, \Omega_2, \cdots, \Omega_n\}$ is a quasi-superior function.*

PROOF. For any point $(x, y) \in (C, S]$ there is at least one index i $(1 \leq i \leq n)$ such that $\Omega(x,y) = \Omega_i(x,y)$. $\overline{\Box}\Omega(x,y) \leq \overline{\Box}\Omega_i(x,y) \leq f(x,y,\Omega_i(x,y)) = f(x, y, \Omega(x, y))$ shows that $\Omega(x, y)$ is quasi-superior. Q.E.D.

7. Ψ_β-, Φ_β-functions. *Let $\beta(x, y)$ be a bounded function defined on C.*

DEFINITION. *We call $\psi(x, y)$ Ψ_β-function on $[C, S]$ if*

i) *$\psi(x, y)$ is quasi-superior with respect to (E_1) on $[C, S]$,*

ii) *$\psi(x, y) \geq \beta(x, y)$ on C.*

We call $\varphi(x, y)$ Φ_β-function on $[C, S]$ if

i) *$\varphi(x, y)$ is quasi-inferior with respect to (E_1) on $[C, S]$,*

ii) $\varphi(x, y) \leq \beta(x, y)$ on C.

We write simply $\psi \in \Psi_\beta$, and $\varphi \in \Phi_\beta$.

REMARK 1. It is not always possible to find a Ψ_β-function or a Φ_β-function. If it is possible to find at least one Ψ_β- and one Φ_β-function, we say that *the condition (P) is satisfied for (E_1) and $\beta(x, y)$.*

REMARK 2. If $f(x, y, u)$ is bounded on $(C, S] \times (-\infty, \infty)$ the condition (P) is automatically satisfied. Indeed, if $|f(x, y, u)| \leq M$, put

$$\psi(x, y) = \Gamma + M(a^2 - (x^2 - 2y))/4,$$
$$\varphi(x, y) = \gamma - M(a^2 - (x^2 - 2y))/4,$$

where γ, Γ, a are the constants such that $\gamma \leq \beta(x, y) \leq \Gamma$ on C, and $a^2 - (x^2 - 2y) > 0$ on $[C, S]$. Then we have

$$\overline{\Box}\psi(x, y) = -M \leq f(x, y, \psi(x, y)),$$
$$\underline{\Box}\varphi(x, y) = M \geq f(x, y, \varphi(x, y)), \quad (x, y) \in (C, S]$$

and

$$\varphi(x, y) \leq \gamma \leq \beta(x, y) \leq \Gamma \leq \psi(x, y) \quad (x, y) \in C.$$

THEOREM 7.1. *If $\psi(x, y)$ is a Ψ_β-function and $\varphi(x, y)$ is a Φ_β-function on $[C, S]$, then $\varphi(x, y) \leq \psi(x, y)$.*

PROOF. $\varphi \leq \psi$ on C is evident by the definition. By Theorem 2.3 we have $\varphi \leq \psi$ on $(C, S]$. Q.E.D.

THEOREM 7.2. *If $\psi_1, \psi_2, \cdots, \psi_n \in \Psi_\beta$; $\varphi_1, \varphi_2, \cdots, \varphi_n \in \Phi_\beta$, then*

$$\psi = \mathrm{Min}\{\psi_1, \psi_2, \cdots, \psi_n\}$$
$$\varphi = \mathrm{Max}\{\varphi_1, \varphi_2, \cdots, \varphi_n\}$$

belong to Ψ_β and Φ_β respectively.

DEFINITION. *Let (C, S) be a p-domain and $(\mathcal{L}, S')$ be its C^1-p-subdomain such that $S' \subseteq S$. For any continuous function $u(x, y)$ on $[C, S]$ we set*

$$(7.1)\qquad M_{\mathcal{L}}u(x, y) = \begin{cases} \text{solution of } (E_1) \text{ which is continuous on } [\mathcal{L}, S'] \text{ and admits the boundary value } u(x, y) \text{ on } \mathcal{L} & (x, y) \in [\mathcal{L}, S'], \\ u(x, y) & (x, y) \in [C, S] - [\mathcal{L}, S']. \end{cases}$$

REMARK. It is not always possible to construct $M_{\mathcal{L}}u(x, y)$ since the continuous solution of (E_1) on $[\mathcal{L}, S']$ does not necessary exist. But if $w(\mathcal{L}, S')$ or $h(\mathcal{L}, S')$ is sufficiently small, or $f(x, y, u)$ is quasi-bounded with respect to u, it is possible to define $M_{\mathcal{L}}u(x, y)$. Indeed, we can first find a continuous solution of $\Box u = 0$ on $[\mathcal{L}, S']$ with the boundary value $u(x, y)$ on $\mathcal{L}$ and next find a solution of $\Box v = f(x, y, v + h(x, y))$ which is continuous on $[\mathcal{L}, S']$ and which vanishes on $\mathcal{L}$.

THEOREM 7.3. *If $w(x, y)$ is quasi-superior with respect to (E_1) on $[C, S]$, then $M_{\mathcal{L}}w(x, y)$ is also quasi-superior.*

PROOF. i) If $(x, y) \in [\mathcal{L}, S']$, then $M_{\mathcal{L}}w = w$ and

$$(7.2)\qquad \overline{\Box} M_{\mathcal{L}}w(x, y) \leq f(x, y, M_{\mathcal{L}}w(x, y)).$$

ii) If $(x, y) \in (\mathcal{L}, S']$, since $M_{\mathcal{L}}w$ is a solution of (E_1) in $(\mathcal{L}, S']$,

$$\Box M_{\mathcal{L}}w(x, y) = f(x, y, M_{\mathcal{L}}w(x, y)).$$

iii) If (x, y) is on the lower bounding segment of the C^1-p-domain $(\mathcal{L}, S']$, $M_{\mathcal{L}}w = w$ in this case also, so (7.2) holds true.

iv) If (x, y) is on the side boundary curve of $(\mathcal{L}, S']$, since a quasi-superior function is a majorant function, $M_{\mathcal{L}}w(\xi, \eta) \leq w(\xi, \eta)$ where $\xi = x + \sqrt{2}\, r \sin\theta \sqrt{\log \operatorname{cosec}^2\theta}$, $\eta = y - r^2 \sin^2\theta$, and from the expression

$$M_{\mathcal{L}}w(\xi, \eta) - M_{\mathcal{L}}w(x, y) = M_{\mathcal{L}}w(\xi, \eta) - w(x, y) \leq w(\xi, \eta) - w(x, y)$$

we have

$$\overline{\Box} M_{\mathcal{L}}w(x, y) \leq \overline{\Box} w(x, y) \leq f(x, y, w(x, y)) = f(x, y, M_{\mathcal{L}}w(x, y)).$$

Q.E.D.

THEOREM 7.4. *If $\psi(x,y)$ is a Ψ_β-function on $[C, S]$, then $M_{\mathcal{L}}\psi(x, y)$ is also a Ψ_β-function.*

Analogous theorems hold true for quasi-inferior functions and Φ_β-functions.