146. Note on a Theorem for Metrizability

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In the present note, we shall apply the previous metrization theorem[1] to an open problem and shall prove the metrizability of a T_1-space X satisfying the following condition of T. Inagaki:[2]

For every point p of X, we can assign a nbd (=neighborhood) basis $\{V_n(p) \mid n=1, 2\cdots\}$ such that

I) *for every $p \in X$ and n, there exists $m=\alpha(p, n)$ such that $p \in V_m(q)$ implies $V_m(q) \subseteq V_n(p)$,*

II) *for every $p \in X$ and n, there exists $l=\beta(p, n)$ such that $q \in V_l(p)$ implies $p \in V_n(q)$.*

Theorem. *In order that a T_1-space X is metrizable it is necessary and sufficient that X satisfies the above condition.*

Proof. Since the necessity is clear, we prove only the sufficiency.

1. First, we remark that we can assume, without loss of generality, that $m<n$ implies $V_m(p) \supseteq V_n(p)$ for every $p \in X$; otherwise we have the fulfilment of the condition by replacing $V_n(p)$ with $V_1(p) \frown \cdots \frown V_n(p)$.

2. For every $p \in X$ and n, we can choose $k=\gamma(p, n)$ such that $q \in V_k(p)$ implies $p \in V_m(q) \subseteq V_n(p)$ for $m=\alpha(p, n)$.

To show this, let $m=\alpha(p, n)$, $k=\beta(p, m)=\gamma(p, n)$. Then $q \in V_k(p)$ implies $p \in V_m(q) \subseteq V_n(p)$ by I) and II).

3. For every $p \in X$ and n, there exist nbds $M_n^1(p)$ and $M_n^2(p)$ of p such that $q \notin V_n(p)$ implies $M_n^1(p) \frown M_n^2(q)=\phi$.

We let $\quad k=\gamma(p, n), \quad l=\beta(p, n), \quad k'=\gamma(p, l);$

$$V_k(p)=M_n^1(p), \qquad V_{k'}(p)=M_n^2(p).$$

Now, let $q \notin V_n(p)$, $r \in M_n^1(p) \frown M_n^2(q) \neq \phi$.

Then in the case of $m=\alpha(p, n) \leqq \alpha(q, l)=m'$,[3] we have

$$q \in V_{m'}(r) \subseteq V_m(r) \subseteq V_n(p)$$

from 2, which contradicts $q \notin V_n(p)$.

In the case of $m=\alpha(p, n) \geqq \alpha(q, l)=m'$, we have

$$p \in V_m(r) \subseteq V_{m'}(r) \subseteq V_l(q),$$

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1) J. Nagata: A theorem for metrizability of a topological space, Proc. Japan Acad., **33**, no. 3 (1957), Theorem 1. See, also, J. Nagata: A contribution to the theory of metrization, Jour. Inst. Polytech., Osaka City Univ., **8**, no. 2 (1957).

2) T. Inagaki: Sur les espaces à structure uniforme, Jour. of the Faculty of Sciences, Hokkaido University, **10** (1943). Prof. Inagaki proved in the paper that a separable space satisfying this condition was perfectly separable. We have learned from Prof. K. Morita that the metrization of such a space is an open problem.

3) We remark that this l does not mean $\beta(p, n)$ but $\beta(q, n)$.

which contradicts $q \notin V_n(p)$ by II). Therefore we conclude $M_n^1(p) \frown M_n^2(q) = \phi$.

4. For every $p \in X$ and n, there exist nbds $N_n^1(p)$ and $N_n^2(p)$ such that $q \in N_n^1(p)$ implies $N_n^2(q) \subseteq U_n(p) = S(p, \mathfrak{V}_n)$,[4] where $\mathfrak{V}_n = \{V_n(x) \mid x \in X\}$.

To show this, we let

$$j = \gamma'(p, n) = \mathrm{Max}\,(\alpha(p, n), \gamma(p, n)).$$

Then we see from 2 that $p \in V_j(q)$ implies $V_j(q) \subseteq V_n(p)$ and that $q \in V_j(p)$ implies $V_j(q) \subseteq V_n(p)$. Next, we let

$$N_n^1(p) = S(p, \mathfrak{V}_n'), \text{ where } \mathfrak{V}_n' = \{V_j(x) \mid j = \gamma'(x, n), x \in X\},$$

$N_n^2(p) = V_j(p)$, where $j = \gamma'(p, n)$. If $q \in N_n^1(p)$, then there exists $x \in X$ satisfying $p, q \in V_j(x)$, $j = \gamma'(x, n)$.

In the case of $j = \gamma'(x, n) \leqq \gamma'(q, n) = j'$, we have

$$N_n^2(q) = V_{j'}(q) \subseteq V_j(q) \subseteq V_n(x)$$

from $q \in V_j(x)$. Since $p \in V_n(x)$ is clear, this implies

$$U_n(p) = S(p, \mathfrak{V}_n) \supseteq V_n(x) \supseteq N_n^2(q).$$

In the case of $j = \gamma'(x, n) \geqq \gamma'(q, n) = j'$, we have

$$p \in V_j(x) \subseteq V_{j'}(x) \subseteq V_n(q)$$

from $q \in V_{j'}(x)$. Hence $U_n(p) = S(p, \mathfrak{V}_n) \supseteq V_n(q)$, and hence

$$N_n^2(q) = V_{j'}(q) \subseteq V_n(q) \subseteq U_n(p).$$

5. Using the result of 3, we let

$$\mathfrak{M}_n = \{M_n^1(x) \mid x \in X\}; \quad S(p, \mathfrak{M}_n) = T_n^1(p), \quad M_n^2(p) = T_n^2(p).$$

Then $q \notin U_n(p)$ $(= S(p, \mathfrak{V}_n))$ implies $T_n^2(q) \frown T_n^1(p) = \phi$. Using the result of 4, we let

$$S_n^1(p) = N_n^1(p) \frown T_n^1(p), \quad S_n^2(p) = N_n^2(p) \frown T_n^2(p).$$

Then we have nbds $U_n(p)$, $S_n^1(p)$, $S_n^2(p)$ for every point p of X such that

i) $\{U_n(p) \mid n = 1, 2 \cdots\}$ is a nbd basis of p,

ii) $q \notin U_n(p)$ implies $S_n^2(q) \frown S_n^1(p) = \phi$,

iii) $q \in S_n^1(p)$ implies $S_n^2(q) \subseteq U_n(p)$.

Therefore we conclude the metrizability of X by our previous theorem.[5]

Corollary. *In order that a T_1-space X is metrizable it is necessary and sufficient that we can assign a sequence $\{V_n(p) \mid n = 1, 2 \cdots\}$ of nbds of every point p of X such that $\bigcap_{n=1}^{\infty} (\bigcup_{p \in A} V_n(p)) = \bigcap_{n=1}^{\infty} S(A, \mathfrak{V}_n) = \bar{A}$[6] for every subset A of X, where $\mathfrak{V}_n = \{V_n(p) \mid p \in X\}$.*

Proof. The necessity is obvious. Let X be a T_1-space satisfying this condition and let $U(p)$ be an arbitrary nbd of an arbitrary point p of X, then $p \notin \overline{X - U(p)}$. Hence $p \notin S(X - U(p), \mathfrak{V}_n)$ for some n, i.e. $S(p, \mathfrak{V}_n) \subseteq U(p)$. Hence $\{V_n(p) \mid n = 1, 2 \cdots\}$ is a nbd basis of p. Since

4) $S(p, \mathfrak{V}_n) = \bigcup \{V \mid p \in V \in \mathfrak{V}_n\}$.

5) Loc. cit.

6) $S(A, \mathfrak{V}_n) = \bigcup \{V \mid V \frown A \neq \phi, V \in \mathfrak{V}_n\}$.

$V_n(p)$ is a nbd of p, we can assign $m=\alpha(p, n)$ such that $S(p, \mathfrak{B}_m) \subseteq V_n(p)$. Then $p \in V_m(q)$ implies $V_m(q) \subseteq V_n(p)$.

To show the fulfilment of II) by $\{V_n(p)\}$, we assume the contrary: $q_i \in V_i(p)$, $p \notin V_n(q_i)$ $(i=1, 2\cdots)$ for definite $p \in X$ and n. Let $A=\{q_i \mid i=1, 2\cdots\}$, then $p \in \bar{A}$ and $p \notin \cup\{V_n(q) \mid q \in A\}$, which is a contradiction. Thus we conclude the validity of this corollary by the theorem.