144. On a Theorem on Function Space of A. Grothendieck

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Very interesting results on countable compactness of function spaces have been obtained by A. Grothendieck [3]. In this paper, we shall consider the case of a set of real valued continuous functions on a pseudo-compact space, and we shall give a generalisation of his result. Let E be a pseudo-compact space, and let $C_s(E)$ be *the topological space of all continuous functions on E with simple convergence topology.* Then *if a subset A of* $C_s(E)$ *is conditionally compact, then it is conditionally countably compact.* Next, *if A is conditionally countably compact, for every sequence* $\{f_m\}$ *of A and countable set* $\{x_m\}$ *of E, there is a continuous function* $f(x)$ *on the closure C of* $\{x_n\}$ *such that, for every x of C,* $f(x)$ *is a cluster point of* $f_n(x)$ *and* $\{f_n(x)\}$ *is pointwise bounded.*

Let $\{f_m\}$ be a countable set of $C_s(E)$, and let $\{x_n\}$ be a countable set of E. Then following A. Grothendieck [3] we shall define *a double cluster point* α *of the double sequence* $\{f_m(x_n)\}$ as follows.

A point (number) α is said to be a *double cluster point* of $\{f_m(x_n)\}$, if, for each neighbourhood U of α, and a given integer N, there are infinitely many $f_m(x_n)$ meeting U for $m, n\geq N$.

If $\{f_m\}$ *and* $\{x_n\}$ *satisfy the conditions in the previous section, then* $\{f_m(x_n)\}$ *has at least one double cluster point.* To prove it, we shall define an equivalent relation on E. For x, y of E, we define $\rho(x, y)$ by

$$\rho(x, y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|f_n(x)-f_n(y)|}{1+|f_n(x)-f_n(y)|}$$

(for example, see R. G. Bartle [1, p. 48]).

By $\rho(x,y)=0$ we shall define an equivalent relation $x\sim y$. Then the space E is decomposed into equivalent classes by the relation "$\sim$". By $\mathfrak{E}$, we denote the set of equivalent classes. Then $\mathfrak{E}$ is a metric space with the metric ρ. By the continuity of the natural mapping $F: E\rightarrow\mathfrak{E}$, $\mathfrak{E}$ is a compact metric space with respect to ρ and $X^\rho=F(x)$ is the class containing x. For $f\in C_s(E)$, we shall define φ on $\mathfrak{E}$ by $\varphi(X^\rho)=f(x)$, where $X^\rho=F(x)$. Therefore, for f_m and f, we have continuous functions $\varphi_n(X^\rho)$, $\varphi(X^\rho)$ on X^ρ for $x\in C$. Since $\mathfrak{E}$ *is compact,*[1] the set X^ρ for x of C has at least one cluster point X_0^ρ. Therefore let α be $\varphi(X_0^\rho)$, then we obtain that α is a double cluster of $f_m(x_n)$. It is sufficient to show that $\varphi(X_0^\rho)$ is a double cluster point of $\varphi_m(X_n^\rho)$, where $X_n^\rho=F(x_n)$. Since X_0^ρ is a cluster point of $\{X_n^\rho\}$, there is a subsequence $\{X_{n_i}^\rho\}$ which

1) See. K. Iséki [5, p. 424].

converges to X_0^p, we have $\varphi_m(X_{n_i}^p) \to \varphi_m(X_0^p)$ for every m.

On the other hand, since $\varphi(X^p)$ is continuous, $\varphi_m(X^p)$ is simply uniformly convergent to $\varphi(X^p)$. Hence, for every $\varepsilon>0$ and every M, there is an integer $m \geq M$ and a neighbourhood U of X_0^p such that

$$|\varphi_m(X^p)-\varphi(X^p)|<\varepsilon$$

for $X^p \in U$. Therefore, for infinitely many n, we have

$$|\varphi_m(X_{n_i}^p)-\varphi(X_{n_i}^p)|<\varepsilon.$$

Since $\varphi(X_{n_i}^p) \to \varphi(X_0^p)$, we have $\varphi_m(X_{n_i}^p) \to \varphi(X_0^p)$ for $i \to \infty$. Next we must show that there are infinitely many n and for the every fixed n, each neighbourhood of $\varphi(X_0^p)$ meets infinitely many of $\varphi_m(X_n^p)$. For a given $\varepsilon>0$, we can find a neighbourhood U of X_0^p such that $|\varphi(X^p)-\varphi(X_0^p)| <\varepsilon$ for $X^p \in U$. Since U contains infinitely many of X_n^p, we shall take one point $X_{n_i}^p$ of $\{X_n^p\}$. Then we can take an integer M such that $M \leq m$ implies

$$|f_m(X_{n_i}^p)-f(X_{n_i}^p)|<\varepsilon$$

and, hence, we have

$$|\varphi_m(X_{n_i}^p)-\varphi(X_0^p)| \leq |\varphi_m(X_{n_i}^p)-\varphi(X_{n_i}^p)|+|\varphi(X_{n_i}^p)-\varphi(X_0^p)|<2\varepsilon$$

for $M \leq m$. Therefore, the proof is complete.

We shall show that *the condition concluded and the pointwise boundedness imply the conditionally compactness of A*. To prove it, we use the technique of F. Eberlein.[2] The available method was also used by A. Grothendieck ([3, p. 173] or [4, p. 19]). Therefore, our proof is similar with them. For each x, let l_x be $\max_{f \in A}|f(x)|$, then l_x is finite. Next consider the product space $\prod_{x \in E}[-l_x, l_x]$ with weak topology, where $[-l_x, l_x]$ denotes the interval $\{y \mid -l_x \leq y \leq l_x\}$. By Tychonov theorem, the product space is compact, and A is considered as a subset of the space. We shall prove that an element of the closure $\bar{A}$ of A is a continuous function.[3] Suppose that $f(x)$ is a non-continuous function of $\bar{A}$, then there is a point x_0 of E such that $f(x)$ is not continuous at x_0. Therefore there is a positive number ε such that for every neighbourhood U of x_0, we can find a point x of U satisfying $|f(x)-f(x_0)| \geq \varepsilon$. We define $f_n(x)$ of A and x_n of E by the following relations recursively.

1) $$|f_n(x_i)-f(x_i)| \leq \frac{1}{n} \quad \text{for} \quad 0 \leq i \leq n-1$$

2) $$|f_i(x_n)-f_i(x_0)| \leq \frac{1}{n} \quad \text{for} \quad 0 \leq i \leq n$$

and

3) $$|f(x_n)-f(x_0)| \geq \varepsilon.$$

This is possible by the hypothesis. Let α be a double cluster point of $\{f_m(x_n)\}$. From these relations, we have $f_n(x_i) \to f(x_i)$ $(n \to \infty)$ for every

2) See, F. Eberlein: Weak compactness in Banach spaces I, Proc. Nat. Acad. Sci. U. S. A., **33**, 51-53 (1947); or A. Grothendieck [4].

3) On convergence, filter, see G. Bruns und J. Schmidt [2].

i, and $f_i(x)\to f_i(x_0)$ $(n\to\infty)$. Hence α is a cluster point of $\{f(x_i)\}$ and $\{f_m(x_0)\}$, and by (1), $f_m(x_0)\to f(x_0)$. Therefore we have $a=f(x_0)$, which contradicts (3). Hence $f(x)$ is continuous at x_0 therefore we have the following

Theorem 1. Let E be a pseudo-compact space, and A a subset of $C_s(E)$, then the following conditions are equivalent:

1) *A is conditionally compact.*

2) *A is conditionally countably compact.*

3) *A is pointwise bounded, and for every sequence f_n of A and every countable set x_n of E, there is a continuous function $f(x)$ on the closure C of $\{x_n\}$ such that, for every x of C, $f(x)$ is a cluster point of $f_n(x)$.*

4) *A is pointwise bounded, and for every sequence f_m of A and for every countable set x_n of E, $f_m(x_n)$ has at least one double cluster point.*

Theorem 2. Let E be a topological space, and let $f_m(x)$ be any sequence of distinct continuous functions on E. Suppose that $f_m(x)$ is pointwise bounded, and for every set of countable points, $\{f_m(x_n)\}$ has at least one double cluster point. Then every continuous function on E is bounded.

Proof. If there is an unbounded continuous positive function $f(x)$, then we can find a sequence $\{x_n\}$ of points such that $f(x_n)\geq n$. Then $f_m(x)=\text{Min}\{m, f(x)\}(m=1, 2,\cdots)$ is a sequence of continuous functions and it is pointwise bounded on E. For $m\leq n$ we have

$$f_m(x_n)=m.$$

If we denote the (m, n)-element by $f_m(x_n)$, we have the following infinite matrix

$$\begin{pmatrix} 1, & 1, & 1, & 1,\cdots \\ *, & 2, & 2, & 2,\cdots \\ *, & *, & 3, & 3,\cdots \\ *, & *, & *, & 4,\cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

As is easily seen, there is no double cluster point in the double sequence. Therefore the proof is complete.

References

[1] R. G. Bartle: On compactness in functional analysis, Trans. Am. Math. Soc., **79**, 35-37 (1955).

[2] G. Bruns und J. Schmidt: Zur Aquivalenz von Moore-Smith Folge und Filtern, Math. Nachr., **13**, 169-186 (1955).

[3] A. Grothendieck: Critères de compacté dans les espaces fonctionnels géréraux, Am. Jour. Math., **74**, 168-186 (1952).

[4] A. Grothendieck: Théorie des Espaces Vectoriels Topologiques, Part II, Chap. V (1954) (mimeographed note).

[5] K. Iséki: Pseudo-compactness and strictly continuous convergence, Proc. Japan Acad., **33**, 424-428 (1957).