139. Remark on Skolem's Theorem Concerning the Imposibility of Characterization of the Natural Number Sequence

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In 1934, Th. Skolem proved the following famous theorem:[1]

"Any *finite* or *enumerable infinite* set $\boldsymbol{M}$ of propositions which are true with respect to the natural number sequence N and can be expressed by closed formulae[2] in the symbolism of the restricted predicate calculus must be true under another interpretation".

Skolem has proved this theorem by constructing a linearly ordered set N^* of individuals, which is not isomorphic to N and makes all of propositions of $\boldsymbol{M}$ true under an interpretation, with equality in its usual meaning. But, of course, the method of construction of N^* is not sufficiently *constructive*; i.e. it is not *finitary*.

On the other hand, in 1929, K. Gödel established the following theorem,[3] named the *completeness theorem* for the restricted predicate calculus:

"Given an *enumerably infinite* (or *finite*) set of formulae of the restricted predicate calculus, if the negation of every conjunction of a finite number of them is unprovable in the predicate calculus, then they are jointly satisfiable in a non-empty domain".

Under the completeness theorem, which is proved by use of non-finitary methods, Skolem's theorem can be easily obtained[4] as a corollary of Gödel's *undecidability theorem*:[5]

"For any consistent *recursive* class κ of axioms, which implies the natural number theory, there exists a recursive predicate $R(*)$, such that the propositions $R(1), R(2), R(3),\cdots$ are all provable from κ but $\forall x R(x)$ is unprovable from κ".

But, in this case, it becomes to be necessary that the set $\boldsymbol{M}$ is, in Gödel's sense, *recursive*.[6]

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1) Über die Nicht-charakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen, Fund. Math., **23**, 150-161 (1934).

2) Formulae containing no free variables are said to be *closed*.

3) Die Vollständigkeit der Axiome des logischen Funktionenkalküls, Monatsh. f. Math. Phys., **37**, 349-360 (1930).

4) Of course, we assume that any class of axioms consisting of only true propositions is consistent.

5) Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, Monatsh. f. Math. Phys., **38**, 173-198 (1931).

6) A class κ of formulae is said to be *recursive*, if and only if the metamathematical relation '$A \in \kappa$' corresponds to a *recursive relation* by the Gödel numbering, where A is a variable expressing an arbitrary formula.

The purpose of the present paper, in relation to the above, is to prove the following

THEOREM. *Let $\boldsymbol{M}$ be a class of axioms and the axiom system $\boldsymbol{M}'$ obtained from $\boldsymbol{M}$ by adjoining all of the propositions*

$$\begin{array}{l} 1=1,\ 1\neq 2,\ 1\neq 3,\cdots, \\ 2\neq 1,\ 2=2,\ 2\neq 3,\cdots, \\ 3\neq 1,\ 3\neq 2,\ 3=3,\cdots, \\ \cdots \end{array}$$

be consistent. And let τ be an individual symbol not occurring in $\boldsymbol{M}$ and $\boldsymbol{M}^*$ be the axiom system obtained from $\boldsymbol{M}$ by adjoining the axioms

$$1\neq\tau,\ 2\neq\tau,\ 3\neq\tau,\cdots .$$

Then $\boldsymbol{M}^*$ is a consistent axiom system.

In the last theorem, the condition 'the cardinal number of $\boldsymbol{M}\leqq\aleph_0$' is unnecessary. And the formal logical system, in connected with which the theorem is concerned, can be arbitrarily chosen, provided that the two following conditions are fulfilled:

1) An axiom system $\boldsymbol{A}$ is inconsistent, if and only if there exist axioms $A_1, A_2,\cdots, A_\mu, B_1, B_2,\cdots, B_\nu$ belonging to $\boldsymbol{A}(\mu, \nu\geqq 0)$ and there exists a formal proposition C and the assertions

$$A_1, A_2,\cdots, A_\mu\rightarrow C$$

and

$$B_1, B_2,\cdots, B_\nu\rightarrow\neg C^{7)}$$

hold;

2) If an assertion

$$F_1(a), F_2(a),\cdots, F_\nu(a)\rightarrow G(a) \quad (\nu\geqq 0)$$

holds, then so is the assertion

$$F_1(n), F_2(n),\cdots, F_\nu(n)\rightarrow G(n),$$

where a is an individual symbol (but is not a bound variable), n is a natural number, and $F_i(n)$ $(i=1, 2,\cdots, \nu)$ or $G(n)$ is the result of substituting n for a throughout $F_i(a)$ or $G(a)$, respectively.

PROOF OF THE THEOREM. If $\boldsymbol{M}^*$ were inconsistent, then there should exist axioms $A_1, A_2,\cdots, A_\mu, B_1, B_2,\cdots, B_\nu$ belonging to $\boldsymbol{M}$ and a proposition $C(\tau)$ and natural numbers $m_1, m_2,\cdots, m_\rho, n_1, n_2, \cdots, n_\sigma$, and the assertions

$$A_1, A_2,\cdots, A_\mu, m_1\neq\tau, m_2\neq\tau,\cdots, m_\rho\neq\tau\rightarrow C(\tau)$$

and

$$B_1, B_2,\cdots, B_\nu, n_1\neq\tau, n_2\neq\tau,\cdots, n_\sigma\neq\tau\rightarrow\neg C(\tau)$$

should hold. Let n be a natural number distinct from $m_1, m_2,\cdots, m_\rho$, $n_1, n_2,\cdots, n_\sigma$, then the assertions

$$A_1, A_2,\cdots, A_\mu, m_1\neq n, m_2\neq n,\cdots, m_\rho\neq n\rightarrow C(n)$$

and

$$B_1, B_2,\cdots, B_\nu, n_1\neq n, n_2\neq n,\cdots, n_\sigma\neq n\rightarrow\neg C(n)$$

7) $\neg C$ is the formal negation of C.

should hold, and it should mean that $\boldsymbol{M}'$ is inconsistent. Hence, $\boldsymbol{M}^*$ is a consistent axiom system, q. e. d.

REMARK. The domain of free or bound individual variables occurring in $\boldsymbol{M}$ is informally N, but in $\boldsymbol{M}^*$ it contains τ also. Accordingly, for example, when $\boldsymbol{M}$ is the usual axiom system of arithmetic and the logical system is the ordinary predicate calculus, the propositions

$$1<\tau,\ 2<\tau,\ 3<\tau, \cdots$$

are all provable from $\boldsymbol{M}^*$.