6. A Generalisation of a Theorem of W. Sierpiński

By Kiyoshi ISÉKI

(Comm. by K. KUNUGI, M.J.A., Jan. 13, 1958)

Some well-known results on the continuum hypothesis by W. Sierpiński have been generalised by the late Professor S. Ruziewicz [1, 2]. In this Note, we shall generalise a recent result of W. Sierpiński [3]. First we shall explain some terminologies needed. By a (closed) *segment* of an ordered set M is meant $\{x \mid a \leq x \leq b,\ x \in M\}$ for a, b of M with $a < b$. We call a and b its *endpoints* of such a segment, and by $[a, b]$ or $[b, a]$, we denote the segment with endpoints a and b. By $\overline{\overline{A}}$, we denote the *power* of A. Then we have the following

Theorem. Let M be an ordered set with cardinal number $\mathfrak{m}$. A cardinal number $\mathfrak{n}$ is not less than $\mathfrak{m}$ if and only if the following proposition holds true: for every element a of M, we can assign a family $\mathfrak{F}(a)$ of interval such that each interval of it has a as endpoint and $\overline{\overline{\mathfrak{F}(a)}} < \mathfrak{n}$, and one of any distinct elements of M is an endpoint of an interval of some $\mathfrak{F}(a)$.

Proof. To prove it, we shall use the idea of Sierpiński [3]. Suppose that $\mathfrak{m} \leq \mathfrak{n}$, and $\mathfrak{m} = \aleph_\alpha$. The ordered set M is well-ordered of type ω_α (ω_α is the initial ordinal of $\aleph_\alpha$): $x_1, x_2, \cdots, x_\omega, \cdots, x_\xi, \cdots$ $(\xi < \omega_\alpha)$. For every x_α $(\alpha < \omega_\alpha)$, we shall consider the family $\mathfrak{F}(a)$ such that $[x_\alpha, x_\xi]$ $(\xi < \alpha)$. Therefore $\overline{\overline{\mathfrak{F}(a)}} < \mathfrak{m} \leq \mathfrak{n}$, hence $\overline{\overline{\mathfrak{F}(a)}} < \mathfrak{n}$. Let x_α, x_β be two distinct elements of M, then we have $\alpha \neq \beta$. If $\alpha < \beta$, then the interval $[x_\alpha, x_\beta]$ is contained in $\mathfrak{F}(b)$ corresponding to β. If $\alpha > \beta$, then $[x_\alpha, x_\beta]$ is contained in $\mathfrak{F}(a)$ corresponding to α.

Conversely, we shall show that the proposition implies $\mathfrak{m} \leq \mathfrak{n}$. To prove it, we shall suppose $\mathfrak{m} > \mathfrak{n}$. Let $\Phi(a)$ be the set of endpoints of $\mathfrak{F}(a)$. For two distinct elements a, b, we have $a \in \Phi(b) - b$ or $b \in \Phi(a) - a$. Let N be a subset of M of cardinal number $\mathfrak{n}$, and let A be the set of the union of $\Phi(a)$ for a of N. Since $\overline{\overline{\Phi(a)}} < \mathfrak{n}$, the cardinal number of A is $\mathfrak{n}$. Therefore we can find an element x of M such that x is not contained in A. Let a be an element of N, then $a \neq x$ and x is not contained in $\Phi(a)$. Therefore we have $a \in \Phi(x)$, and $N \subset \Phi(x)$. This shows that $\overline{\overline{\Phi(x)}} \geq \mathfrak{n}$, which is a contradiction. Hence $\mathfrak{m} \leq \mathfrak{n}$.

References

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[3] W. Sierpiński: Sur une propriété de la droite équivalente à l'hypothèse du continu, Ganita, **5**, 113-116 (1954).