2. Duality in Mathematical Structure

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(Comm. by Z. SUETUNA, M.J.A., Jan. 13, 1958)

1. In many representation theorems of different branches of abstract mathematics, for example, representation of ordered sets by cuts, that of lattices by sets of ideals, the conjugate spaces of Banach spaces, or the duality of groups, there seem to appear some similar conceptions. In order to deal with those representation theorems simultaneously, we made an attempt to set up a concept of a universal mathematical structure, in which the main rôle is played by some selected applications of a system to a system, which we call homomorphisms, but they may be continuous mappings in topological spaces, order-preserving mappings in ordered sets or linear operators in linear spaces.

The definitions of our structure and some results of them will be stated in this note, but we omit the detail of proofs which, as well as the applications to individual specialized systems, will be published elsewhere.

2. Let $\mathfrak{S}$ be a family of sets. A set in $\mathfrak{S}$ is called a *system*. If X and Z are systems and $Z\subset X$, then Z is called a *subsystem* of X. We assume that, to each pair of systems X and Y, a family $\mathrm{Hom}(X, Y)$ of applications which map X into Y is distinguished. An application φ in $\mathrm{Hom}(X, Y)$ is called a *homomorphism* of X in Y. Further we assume that those homomorphisms and the family $\mathfrak{S}$ satisfy the following axioms which fall into five groups (A)–(E).

In these statements of axioms, the letters X, Y and Z denote systems.

The axioms of the group (A) are concerned with the conditions for an application of a system to be a homomorphism.

(A1) *If $Z\subset X$, and I_Z is the identical mapping on Z, then $I_Z\in Hom(Z, X)$.*

(A2) *If $\varphi\in Hom(X, Z)$ and $\psi\in Hom(Z, Y)$, then $\psi\varphi\in Hom(X, Y)$.*

(A3) *If $\varphi\in Hom(X, Y)$ and $\varphi(X)\subset Z$, then $\varphi\in Hom(X, Z)$.*

The axioms of the group (B) are concerned with the conditions for a set to be a system.

(B1) *If $\varphi\in Hom(X, Y)$, then $\varphi(X)\in\mathfrak{S}$.*

(B2) *If $\varphi\in Hom(X, Y)$ and $Z\frown\varphi(X)\neq\phi$, then $\varphi^{-1}(Z)\in\mathfrak{S}$.*[1]

(B3) *There exists a one element system $\{e\}$ such that for any system X the application which maps each element x of X onto e is*

1) $\varphi(X)=\{\varphi(x);\ x\in X\}$, $\varphi^{-1}(Z)=\{x;\ \varphi(x)\in Z\}$.

a homomorphism of X *in* $\{e\}$.[2]

If $\varphi \in \mathrm{Hom}(X, Y)$ and $\varphi(X)=Y$, then φ is called a homomorphism of X *on* Y, and Y is called *homomorphic to* X. A homomorphism φ of X in Y is called an *isomorphism*, if it is one-to-one and its inverse is also a homomorphism of $\varphi(X)$ on X. If there exists an isomorphism of X on Y, then they are called *isomorphic*.

Before stating the remaining axioms we shall introduce the definition of the direct product.[3]

DEFINITION 1. *Let* X_λ; $\lambda \in \Lambda$, *where* Λ *is a set of indices, be a family of systems. A system* W, *denoted by* $\prod_\lambda X_\lambda$, *is called the direct product of* X_λ*'s, if* W *satisfies the following conditions*:

i) W *is a set-product of* X_λ*'s, that is, there exists a one-to-one correspondence between an element* w *of* W *and a sequence* $\{x_\lambda\}$ *of elements* $x_\lambda \in X_\lambda$. *This correspondence will be denoted by* $w=\pi_\lambda\{x_\lambda\}$ *and* $x_\lambda=p_\lambda(w)$, *where* w *corresponds to* $\{x_\lambda\}$.

ii) $p_\lambda \in Hom(W, X_\lambda)$ *for each* $\lambda \in \Lambda$.

iii) *If* Z *is a system and* $\varphi_\lambda \in Hom(Z, X_\lambda)$ *for each* $\lambda \in \Lambda$, *then the mapping* φ *which is defined by* $\varphi(z)=\pi_\lambda\{\varphi_\lambda(z)\}$ *for* $z \in Z$ *is a homomorphism of* Z *in* W.

Especially if the set Λ is finite and $\Lambda=\{1, 2, \cdots, n\}$, then we may use the following notation: $X_1 \times X_2 \times \cdots \times X_n = \prod_\lambda X_\lambda$. If each X_λ is a same set X, then X^Λ is defined as $\prod_\lambda X_\lambda$.

Now we shall give a further axiom (C):

(C) *For any family* $\{X_\lambda; \lambda \in \Lambda\}$ *of systems, where* Λ *is a set of indices, there exists a system which is the direct product* $\prod_\lambda X_\lambda$.[4]

It is easily seen that if both U and W are direct products of X_λ's, then U and W are isomorphic.

Next, let X and Y be any systems. We put $Y_x=Y$ for each $x \in X$ and make the direct product Y^X. Then Y^X consists of all applications from X in Y. Hence $\mathrm{Hom}(X, Y)$ is considered as a subset of Y^X. Under this interpretation we give the fourth axiom (D).

(D) *Hom* (X, Y) *is a subsystem of* Y^X.[5]

The family $\mathfrak{S}$ may be a very vast one. For example it may be a family of whole topological spaces. Here, to avoid some pathology

2) It seems to be convenient in many investigations to add the following axiom: (B4) *If* X_λ, $\lambda \in \Lambda$, *are systems and* $\cap X_\lambda \neq \phi$, *then* $\cap X_\lambda$ *is a system.*

But this axiom is not necessary in this paper.

3) This definition and Theorem 1 are stated by V. S. Krishnan: Closure operations on c-structure, Indagationes Math., **15**, 317-329 (1953).

4) For example, the family of all totally ordered sets does not satisfy this axiom. In Banach spaces, the definition of direct products must be modified.

5) The set of all representations of a group by n-square matrices does not make itself a group, and our theorems can not be applied to Tannaka's duality theorem.

of the set-theory, we assume

(E) *For any subfamily* $\mathfrak{T}$ *of* $\mathfrak{S}$ *such that each system in* $\mathfrak{T}$ *has a power less than some cardinal number, a family* $\mathfrak{B}$ *of systems can be selected in such a way that this family* $\mathfrak{B}$ *makes itself a set with some power, and any system in* $\mathfrak{T}$ *is isomorphic to one of the systems in* $\mathfrak{B}$.

3. Let φ and ψ be homomorphisms of a system X. We say that φ and ψ are equivalent if there exists an isomorphism of $\varphi(X)$ on $\psi(X)$ with $\psi=\theta\varphi$. Let Hom (X) be the family of all homomorphisms of X. If we identify equivalent homomorphisms in Hom (X), then by the axiom (E), Hom (X) makes itself a set with some power.

If we introduce an order $\leqq$ between homomorphisms in Hom (X) in such a way that $\varphi\leqq\psi$ implies that there exists a $\theta\in$ Hom $(\psi(X), \varphi(X))$ with $\varphi=\theta\psi$, then Hom (X) becomes a partially ordered set. But by the definition of the direct product and the axioms (C) and (B3) we have

THEOREM 1. *Hom* (X) *is a complete lattice by the order* $\leqq$.

Let Φ be a subset of Hom (X), and let $\vee\Phi$ denote the least upper bound of homomorphisms in Φ, then we can easily see

LEMMA 1. *Put* $\psi=\vee\Phi$. *We have* $\psi(x_0)=\psi(x_1)$, $x_0, x_1\in X$, *if and only if* $\varphi(x_0)=\varphi(x_1)$ *for any* $\varphi\in\Phi$.

Hereafter we fix a system L, and Hom (X, L) is denoted by X^*. The first main theorem which is deduced from the axioms (A) (B) (C) (D) and (E) is the following

THEOREM 2. (1) *Let* X *be a system. If we put* $\dot{x}(\varphi)=\varphi(x)$ *for* $x\in X$ *and* $\varphi\in X^*$, *then* $\dot{x}\in X^{**}=Hom\,(X^*, L)$.

(2) *If we put* $\tau(x)=\dot{x}$, *then* $\tau\in Hom\,(X, X^{**})$.

(3) τ *is equivalent to* $\vee\, Hom\,(X, L)$.

(4) *The necessary and sufficient condition that we have* $I_X=\vee\, Hom\,(X, L)$[6] *is that for a sufficiently large set* Λ *of indices,* X *is isomorphically embedded in* L^{Λ}.

Similarly, by putting $\dot{\varphi}(\xi)=\xi(\varphi)$ for $\varphi\in X^*$ and $\xi\in X^{**}$, we have a homomorphism $\dot{\varphi}\in X^{***}$, and putting $\tau^\circ(\varphi)=\dot{\varphi}$, we have a homomorphism $\tau^\circ\in$ Hom (X^*, X^{***}). But by the definition of Hom (X, L) $=X^*$, X^* is a subsystem of L^X, and hence by Theorem 2 (3) and (4), τ° is an isomorphism. But further we have

THEOREM 3. (1) *Let* η *be a homomorphism in* $X^{***}=Hom\,(X^{**}, L)$. *Put* $\varphi_\eta(x)=\eta(\dot{x})$, *for* $x\in X$, ($\dot{x}=\tau(x)$, *see Theorem 2. (1)), then we have* $\varphi_\eta\in X^*=Hom\,(X, L)$.

(2) *Put* $\chi(\eta)=\varphi_\eta$, *then we have* $\chi\in Hom\,(X^{***}, X^*)$.

(3) *The contraction of* χ *on* $\tau^\circ(X^*)$ *is the inverse mapping of the isomorphism* τ°.

6) I_X is the identical mapping on X, see axiom (A1).

4. By the axiom (A2), for any $\theta \in \mathrm{Hom}(X, Z)$ and $\varphi \in \mathrm{Hom}(Z, Y)$, we have a homomorphism $\varphi\theta \in \mathrm{Hom}(X, Y)$. Here we have

THEOREM 4. (1) *Put* $^*\varphi(\theta)=\varphi\theta$, *then we have* $^*\varphi \in Hom(Hom(X, Z), Hom(X, Y))$.

(2) *Put* $\theta^*(\varphi)=\varphi\theta$, *then we have* $\theta^* \in Hom(Hom(Z, Y), Hom(X, Y))$.

(3) *The mapping* $\varphi \to {}^*\varphi$ *is a homomorphism of* $Hom(Z, Y)$ *in* $Hom(Hom(X, Z), Hom(X, Y))$.

(4) *The mapping* $\theta \to \theta^*$ *is a homomorphism of* $Hom(X, Z)$ *in* $Hom(Hom(Z, Y), Hom(X, Y))$.

(5) *If* θ *is a homomorphism of* X *on* Z, *then* θ^* *is an isomorphism of* $Hom(Z, Y)$ *in* $Hom(X, Y)$.

(6) *If* θ *is an isomorphism of* X *in* Z, *and if any homomorphism of* $\theta(X)$ *in* Y *can be extended to a homomorphism of* Z *in* Y, *then* θ^* *is a homomorphism of* $Hom(Z, Y)$ *on* $Hom(X, Y)$.

Now we shall give one more definition:

DEFINITION 2. *A subsystem* Φ *of* $Hom(Z, Y)$ *is called admissible on* Z, *if the mapping which maps the pair* $\langle\varphi, z\rangle$ *onto* $\varphi(z)$ *where* $\varphi \in Hom(Z, Y)$ *and* $z \in Z$ *is a homomorphism of the direct product* $\Phi \times Z$ *in* Y.

We add a proposition in Theorem 4.

THEOREM 4. (7) *Let* Φ *be a subsystem of* $Hom(Z, Y)$. *Put* $^*\Phi=\{^*\varphi; \varphi \in \Phi\}$ *where* $^*\varphi$ *is a homomorphism of* $Hom(X, Z)$ *in* $Hom(X, Y)$ *(see Theorem 4 (3)). If* Φ *is admissible on* Z, *then* $^*\Phi$ *is admissible on* $Hom(X, Z)$.

The statement (4) in Theorem 4 is especially interested. Really by putting $Y=L$, we have the

COROLLARY. *To each homomorphism* θ *in* $Hom(X, Z)$, *there corresponds homomorphically a homomorphism* θ^* *in* $Hom(Z^*, X^*)$.

This proposition gives us a conception similar to the conjugate operator in Banach spaces.[7] Statements (5) and (6) in Theorem 4 are similarly paraphrased.

If θ is a homomorphism in $\mathrm{Hom}(X, Z)$, then θ^* is a homomorphism in $\mathrm{Hom}(Z^*, X^*)$, and similarly we have a homomorphism θ^{**} in $\mathrm{Hom}(X^{**}, Z^{**})$. Here we have

LEMMA 2. *If* $\theta(x)=z$, *then* $\theta^{**}(\dot{x})=\dot{z}$ ($\dot{x}=\tau(x)$, *see Theorem 2*).

Finally we add

THEOREM 5. *For any systems* X, Y *and* Z, $Hom(X, Hom(Z, Y))$ *is isomorphic to* $Hom(Z, Hom(X, Y))$.

$\mathrm{Hom}(X, \mathrm{Hom}(Z, Y))$ can be regarded as the set of all functions $\xi(x, z)$ into Y with two arguments $x \in X$ and $z \in Z$ such that for a fixed

7) In Banach spaces, our definition of the direct product does not hold, and in applying our theorems on Banach spaces, considerable modifications are necessary.

$x \in X$, they are homomorphisms of Z in Y and, for a fixed $z \in Z$, they are homomorphisms of X in Y. Such a function $\xi(x, z)$ is, so to speak, a *bi-homomorphism* similar to the bi-linear operator in linear spaces. For example, $\xi:(\varphi, x) \rightarrow \varphi(x)$ where $\varphi \in \mathrm{Hom}(X, Y)$ and $x \in X$ is a bi-homomorphism of $\mathrm{Hom}(X, Y)$ and X in Y, and the function $\xi:(\varphi, \psi) \rightarrow \psi\varphi$, where $\varphi \in \mathrm{Hom}(X, Z)$ and $\psi \in \mathrm{Hom}(Z, Y)$, is a bi-homomorphism of $\mathrm{Hom}(X, Z)$ and $\mathrm{Hom}(Z, Y)$ in $\mathrm{Hom}(X, Y)$ (see Theorem 4). Theorem 5 is useful in dealing with such functions. Especially putting $Y=L$ and $Z=X^*$ in Theorem 5 we have the

COROLLARY. *Hom* (X, X^{**}) *is isomorphic to Hom* (X^*, X^*).

This corollary seems useful when X is reflexive, that is, if $X=X^{**}$, or when we discuss about the reflexibility of X.