26. Note on Idempotent Semigroups. III

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§1. This note is an abstract of the paper (Naoki Kimura [1]), in which the author proved the structure theorems of some special idempotent semigroups. Terminologies in the previous papers (Kimura [2] and Miyuki Yamada and Kimura [3]) will be used without definitions.

§2. An idempotent semigroup is called

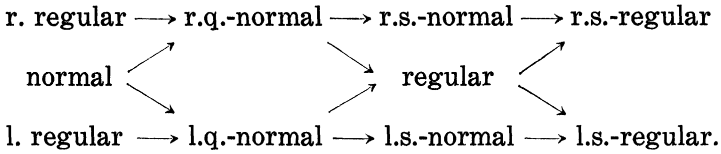
(1) *right* (*left*) *semi-regular* if $axy=axyayxy$ $(xya=xyxaxya)$,

(2) *right* (*left*) *semi-normal* if $axy=axyay$ $(xya=xaxya)$,

(3) *right* (*left*) *quasi-normal* if $axy=axay$ $(xya=xaya)$,

for all a, x, y.

Then the following implications are easy to prove:



Further, we have the following lemmas:

LEMMA 1. *An idempotent semigroup is regular if and only if it is both left and right semi-regular.*

LEMMA 2. *An idempotent semiproup is normal if and only if it is both left and right quasi-normal* (*semi-normal*).

§3. Let $\mathfrak{P}(\mathfrak{Q})$ be the equivalence on an idempotent semigroup S defined by

$$x\mathfrak{P}y \text{ if and only if } xy=y \text{ and } yx=x,$$
$$x\mathfrak{Q}y \text{ if and only if } xy=x \text{ and } yx=y.$$

Then we have the following representation theorems.

THEOREM 1. *$\mathfrak{P}(\mathfrak{Q})$ is a congruence on an idempotent semigroup S if and only if S is left* (*right*) *semi-regular. Further, in this case the quotient semigroup $S/\mathfrak{P}(S/\mathfrak{Q})$ is left* (*right*) *regular.*

From this theorem and Lemma 1, we have

COROLLARY. *Both $\mathfrak{P}$ and $\mathfrak{Q}$ are congruences on an idempotent semigroup S if and only if S is regular. Further, in this case S is isomorphic to the spined product of $S/\mathfrak{P}$ and $S/\mathfrak{Q}$ with respect to its structure semilattice.*

REMARK. This corollary is essentially the same as Theorem 2 in [2], and the above method gives an alternative proof for it.

THEOREM 2. *$\mathfrak{P}(\mathfrak{Q})$ is a congruence on an idempotent semigroup S and $S/\mathfrak{P}(S/\mathfrak{Q})$ is left (right) normal if and only if S is left (right) semi-normal.*

THEOREM 3. *Both $\mathfrak{P}$ and $\mathfrak{Q}$ are congruences on an idempotent semigroup S and $S/\mathfrak{P}(S/\mathfrak{Q})$ is left (right) normal, if and only if S is left (right) quasi-normal. Further, in this case S is the spined product of $S/\mathfrak{P}$ and $S/\mathfrak{Q}$ with respect to its structure semilattice.*

From this theorem and Lemma 2, we have

COROLLARY. *Both $\mathfrak{P}$ and $\mathfrak{Q}$ are congruences on an idempotent semigroup S, $S/\mathfrak{P}$ is left normal and $S/\mathfrak{Q}$ is right normal if and only if S is normal. Further, in this case S is isomorphic to the spined product of $S/\mathfrak{P}$ and $S/\mathfrak{Q}$ with respect to its structure semilattice.*

REMARK. This corollary is essentially the same as Theorem 4 in [3], and the above method gives an alternative proof for it.

References

[1] Naoki Kimura: The structure of idempotent semigroups (III) (to appear).
[2] Naoki Kimura: Note on idempotent semigroups. I, Proc. Japan Acad., **33**, 642 (1957).
[3] Miyuki Yamada and Naoki Kimura: Note on idempotent semigroups. II, Proc. Japan Acad., **34**, 110 (1958).