25. Note on Idempotent Semigroups. II

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§1. This note is an abstract of the main part of the paper by the authors [2], and also it is the continuation of the first paper (Kimura [1]). Any terminology without definition should be referred to [1].

The purpose of this note is to present the structure theorem of normal idempotent semigroups (for the definition, see below) and some relevant matters.

§2. An idempotent semigroup is called (1) *left normal*, (2) *right normal*, (3) *normal*, respectively, if it satisfies the following corresponding identities:

(1) $abc=acb$,

(2) $bca=cba$,

(3) $abca=acba$.

Now the following lemmas are apparent.

Lemma 1. *If S is (left, right) normal, then S is (left, right) regular.*

Lemma 2. *If S is normal and left (right) regular, then S is left (right) normal.*

§3. Theorem 1. *An idempotent semigroup $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$ is left (right) normal if and only if every S_γ is left (right) singular and there exists a family of functions $\Phi=\{\phi_\beta^\alpha:\alpha\geqq\beta,\alpha,\beta\in\Gamma\}$ satisfying*

(1) $\phi_\beta^\alpha:S_\alpha\rightarrow S_\beta$ *for* $\alpha\geqq\beta$,

(2) ϕ_α^α *is the identity function,*

(3) $\phi_\gamma^\beta\phi_\beta^\alpha=\phi_\gamma^\alpha$ *for* $\alpha\geqq\beta\geqq\gamma$,

and (4) $ab=\phi_{\alpha\beta}^\alpha(a)$ $(ab=\phi_{\alpha\beta}^\beta(b))$ *for* $a\in S_\alpha$, $b\in S_\beta$.

Proof. Let $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$ be left normal. Then it is left regular by Lemma 1. Therefore every S_γ is left singular by Theorem 1 of [1].

Let $\alpha\geqq\beta$, $a\in S_\alpha$, x, $y\in S_\beta$. Then by left normality

$$ax=axy=ayx=ay.$$

Therefore aS_β is reduced to one element which is denoted by $\phi_\beta^\alpha(a)$. Then it is straightforward to prove all the above conditions. This ends the proof of the first part of the theorem. The second part is contained in the following

Theorem 2. *Let Γ be a semilattice. Let $\{S_\gamma:\gamma\in\Gamma\}$ be a disjoint*

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family of sets. Let $\Phi=\{\phi_\beta^\alpha : \alpha \geqq \beta, \alpha, \beta \in \Gamma\}$ *be a family of functions satisfying the conditions* (*1*), (*2*) *and* (*3*) *of Theorem 1. Let* $S=\bigcup\{S_\gamma : \gamma \in \Gamma\}$. *Then under the multiplication defined by* (*4*) *of Theorem 1,* S *becomes a left* (*right*) *normal idempotent semigroup, whose structure decomposition is* $S\sim\sum\{S_\gamma : \gamma \in \Gamma\}$. *Further any left* (*right*) *idempotent semigroup can be obtained by this process up to isomorphism.*

Theorem 3. *A left* (*right*) *normal idempotent semigroup is isomorphic to the direct product of a left* (*right*) *singular semigroup and a semilattice if and only if each function of* Φ *of Theorem 2 is one-to-one onto.*

§ 4. Theorem 4. *An idempotent semigroup is normal if and only if it is the spined product of a left normal idempotent semigroup and a right normal idempotent semigroup.*

Proof. Let S be normal; then it is regular by Lemma 1. Therefore by Theorem 2 of [1], S is the spined product of A and B with respect to Γ, where Γ is the structure semilattice of S and $A(B)$ is a left (right) regular semigroup with Γ as its structure semilattice.

Since $A(B)$ is normal and left (right) regular, it must be left (right) normal by Lemma 2.

Sufficiency is obvious.

Theorem 5. *An idempotent semigroup* $S\sim\sum\{A_\gamma\times B_\gamma : \gamma \in \Gamma\}$ *is normal if and only if there exist two families of functions* $\Phi=\{\phi_\beta^\alpha : \alpha \geqq \beta, \alpha, \beta \in \Gamma\}$, $\Psi=\{\psi_\beta^\alpha : \alpha \geqq \beta, \alpha, \beta \in \Gamma\}$ *satisfying*

(1) $\phi_\beta^\alpha : A_\alpha \to A_\beta$, $\psi_\beta^\alpha : B_\alpha \to B_\beta$, $\alpha \geqq \beta$,

(2) ϕ_α^α *and* ψ_α^α *are the identity functions,*

(3) $\phi_\gamma^\beta \phi_\beta^\alpha = \phi_\gamma^\alpha$, $\psi_\gamma^\beta \psi_\beta^\alpha = \psi_\gamma^\alpha$, $\alpha \geqq \beta \geqq \gamma$,

and (4) $(a, b)(a', b') = (\phi_{\alpha\beta}^\alpha(a), \psi_{\alpha\beta}^\beta(b'))$ *if* $(a, b) \in A_\alpha \times B_\alpha$, $(a', b') \in A_\beta \times B_\beta$.

Proof. Necessity follows from Theorems 1 and 4. Sufficiency is contained in Theorem 7 below.

Theorem 6. *An idempotent semigroup* $S\sim\sum\{S_\gamma : \gamma \in \Gamma\}$ *is normal if and only if there exists a family of functions,* $\Theta=\{\theta_\beta^\alpha : \alpha \geqq \beta, \alpha, \beta \in \Gamma\}$, *such that*

(1) $\theta_\beta^\alpha : S_\alpha \to S_\beta$, $\alpha \geqq \beta$,

(2) θ_α^α *is the identity function,*

(3) $\theta_\gamma^\beta \theta_\beta^\alpha = \theta_\gamma^\alpha$, $\alpha \geqq \beta \geqq \gamma$,

and (4) $ab = \theta_{\alpha\beta}^\alpha(a)\,\theta_{\alpha\beta}^\beta(b)$ *if* $a \in S_\alpha$, $b \in S_\beta$.

Proof. Necessity follows from Theorem 5 above. Sufficiency is contained in Theorem 8 below.

The following two theorems are to Theorem 2 as Theorems 5 and 6 are to Theorem 1. They are proved by direct calculations and we omit the proofs here.

Theorem 7. *Let* Γ *be a semilattice. Let* $\{A_\gamma : \gamma \in \Gamma\}$, $\{B_\gamma : \gamma \in \Gamma\}$ *be two disjoint families of sets. Let* Φ, Ψ *be families of functions satis-*

fying the conditions (1), (2) and (3) of Theorem 5. Then under the multiplication defined by (4) of Theorem 5, $S=\bigcup\{A_\gamma\times B_\gamma:\gamma\in\Gamma\}$ *becomes a normal idempotent semigroup, whose structure decomposition is* $S\sim\sum\{A_\gamma\times B_\gamma:\gamma\in\Gamma\}$.

Theorem 8. *Let* Γ *be a semilattice. Let* $\{S_\gamma:\gamma\in\Gamma\}$ *be a disjoint family of rectangular semigroups. Let* Θ *be a family of functions satisfying the conditions (1), (2) and (3) of Theorem 6. Then under the multiplication defined by (4) of Theorem 6,* $S=\bigcup\{S_\gamma:\gamma\in\Gamma\}$ *becomes a normal idempotent semigroup, whose structure decomposition is* $S\sim\sum\{S_\gamma:\gamma\in\Gamma\}$.

Theorem 9. *A normal idempotent semigroup is isomorphic to the direct product of a rectangular semigroup and a semilattice if and only if each function of* Φ *and* $\Psi(\Theta)$ *of the Theorem 5(6) is one-to-one onto.*

§ 5. Consider the identity

$$(*)\qquad a_1a_2\cdots a_n=a_{p_1}a_{p_2}\cdots a_{p_n},$$

where $(p_1, p_2,\cdots, p_n)$ is a non-trivial permutation of $(1, 2,\cdots, n)$. Then we have the following

Theorem 10. *An idempotent semigroup satisfying* $(*)$ *is*

(1) *normal* *if* $p_1=1,\ p_n=n$,
(2) *left normal* *if* $p_1=1,\ p_n\neq n$,
(3) *right normal* *if* $p_1\neq 1,\ p_n=n$,
(4) *commutative* *if* $p_1\neq 1,\ p_n\neq n$.

References

[1] Naoki Kimura: Note on idempotent semigroups. I, Proc. Japan Acad., **33**, 462 (1957).

[2] Miyuki Yamada and Naoki Kimura: The structure of idempotent semigroups (II) (to appear).