20. On Symmetric Skew Unions of Knots

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Introduction. S. Kinoshita and H. Terasaka introduced the notion of *symmetric unions and symmetric skew unions* of knots and showed that the Alexander polynomial of the symmetric union of a knot is the square of that of the original knot. As regards the symmetric skew union of a knot nothing more is obtained than that its Alexander polynomial $\Delta(x)$ is independent of the winding number. In this note we shall give a more explicit form of $\Delta(x)$ and show especially that this is of the form $\phi(x)\cdot\phi(1/x)$.[1]

1. We shall call a polynomial $f(x)$ *symmetric* (*skew symmetric*) if $f(x)=x^p f(1/x)$ ($f(x)=-x^p f(1/x)$) for a suitable integer p. We shall call the integer $n-m$ the *reduced degree* of a polynomial $f(x)=a_l x^l +\cdots+a_n x^n+\cdots+a_m x^m+\cdots+a_1 x+a_0$ if $a_l=a_{l-1}=\cdots=a_{n+1}=0$, $a_n\neq 0$, $a_m\neq 0$ ($n>m$) and $a_{m-1}=\cdots=a_1=a_0=0$.

Lemma 1. *Let $f(x)$ and $F(x)$ be symmetric polynomials with even reduced degrees and let $g(x)$ and $G(x)$ be skew symmetric polynomials, such that*

$$F(x)=xf(x)+(x-1)g(x)$$
$$G(x)=(1-x)f(x)+g(x).$$

Then, if

$$f(x)=a_n x^n+\cdots+a_m x^m$$
$$g(x)=b_n x^n+\cdots+b_m x^m$$

where $n>m$, and a_n or $b_n\neq 0$ and a_m or $b_m\neq 0$, we have either

$$\text{(I)}\qquad \begin{cases} f(x)=a_n x^n+\cdots+a_{m+1}x^{m+1}+a_m x^m \\ g(x)=b_n x^n+\cdots+b_{m+1}x^{m+1} \end{cases}$$

where $a_n=a_m\neq 0$ and $b_{n-i}=-b_{(m+1)+i}$ ($i=1,2,\cdots,n-(m+1)$), or

$$\text{(II)}\qquad \begin{cases} f(x)=\qquad a_{n-1}x^{n-1}+\cdots+a_m x^m \\ g(x)=b_n x^n+b_{n-1}x^{n-1}+\cdots+b_m x^m \end{cases}$$

where $b_n=-b_m\neq 0$ and $a_{(n-1)-i}=a_{m+i}$ ($i=1,2,\cdots,n-(m+1)$).

Proof. By the conditions

$$F(x)=(a_n+b_n)x^{n+1}+(a_{n-1}+b_{n-1}-b_n)x^n+\cdots+(a_m+b_m-b_{m+1})x^{m+1}-b_m x^m$$
$$G(x)=-a_n x^{n+1}+(a_n+b_n-a_{n-1})x^n+\cdots+(a_{m+1}+b_{m+1}-a_m)x^{m+1}+(a_m+b_m)x^m.$$

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1) This ascertains the result of R. H. Fox and J. W. Milnor [2], for any symmetric (skew) unions of knots may easily be proved to belong to the category of knots considered by them.

Now the following four cases are to be considered:

Case 1. $a_n \neq 0$, $b_m \neq 0$ and $a_m = b_n = 0$. By the symmetricity of $F(x)$ and the skew symmetricity of $G(x)$, we have $a_n = -b_m$ and $a_n = b_m$ respectively, which contradict $a_n \neq 0$. Therefore, the case 1 can not actually occur.

Case 2. $a_m \neq 0$, $b_n \neq 0$ and $a_n = b_m = 0$. We are going to prove that this case is also impossible.

First we have $a_{n-1}=0$. For if $a_{n-1}\neq 0$, by the symmetricity of $f(x)$ we have $a_{n-1}=a_m$, and $n-m-1$ must be even. And since $b_m=0$ and $a_n+b_n\neq 0$ and since the reduced degree of $F(x)$ is assumed to be even, $a_m-b_{m+1}=0$, hence $b_{m+1}=a_m\neq 0$; thus by the skew symmetricity of $g(x)$ $b_n=-b_{m+1}$ By the skew symmetricity of $G(x)$ we must have, since $b_m=0$ and $a_n=0$, either $a_m=a_{n-1}-b_n\neq 0$ or $a_{n-1}-b_n=0$. But the former case contradicts $a_m=a_{n-1}$ and $b_n\neq 0$, and the latter case contradicts $a_m-b_{m+1}=0$, $b_n=-b_{m+1}$ and $a_{n-1}=a_m$. Thus we must have $a_{n-1}=0$.

Also we have $b_{m+1}=0$. For if $b_{m+1}\neq 0$, then by the skew symmetricity of $g(x)$ and $G(x)$ we have $b_{m+1}=-b_n=a_m$ and $b_{n-1}=-b_{m+2}$. Suppose now that $a_{n-2}\neq 0$. Since the reduced degree $n-m-2$ of $f(x)$ is even, the coefficient of x^{m+2} of $F(x)$ is equal to zero: i.e. $a_{m+1}+b_{m+1}-b_{m+2}=0$. And by the skew symmetricity of $G(x)$, $b_{n-1}-a_{n-2}=-a_{m+1}$. But from the above properties, $b_{m+1}=a_m=a_{n-2}=b_{n-1}+a_{m+1}=b_{n-1}+b_{m+2}-b_{m+1}=-b_{m+1}\neq 0$, which is impossible. Hence we must have $a_{n-2}=0$. But from the above properties we have $a_{m+1}=-b_{m+1}=b_{m+2}$. Since $a_{m+1}+b_{m+1}-b_{m+2}=b_{m+1}\neq 0$, we have by the symmetricity of $F(x)$, $b_n=b_{m+1}\neq 0$, which contradicts $b_n=-b_{m+1}$. Thus we have seen that $b_{m+1}=0$.

Now by the symmetricity of $F(x)$ and the skew symmetricity of $G(x)$, we have $b_n=a_m\neq 0$ and $b_n=-a_m$, which are impossible. Thus the case 2 can not actually occur.

Case 3. $a_n=a_m\neq 0$. By the symmetricity of $f(x)$ we have $a_{n-i}=a_{m+i}$ $(i=1,2,\cdots,n-m)$.

We assert that $a_n=a_m+b_m\neq 0$. For if $a_m+b_m=0$, then $a_m=-b_m\neq 0$. Then we have $a_n+b_n\neq 0$, for if $a_n+b_n=0$, we must have $b_n=-a_n=-a_m=b_m$ which contradicts $b_n=-b_m\neq 0$. Moreover we have $a_{m+1}+b_{m+1}\neq 0$. For if $a_{m+1}+b_{m+1}=0$, then by the skew symmetricity of $G(x)$, $a_n=a_{m+1}+b_{m+1}-a_m=-a_m$, which contradicts $a_n=a_m\neq 0$. By the symmetricity of $F(x)$ we have $a_n+b_n=-b_m=a_m$ and $a_{n-1}+b_{n-1}-b_n=a_m+b_m-b_{m+1}$, hence we have $b_n=0$ and $a_{n-1}+b_{n-1}=-b_{m+1}$. Here, in view of $G(x)$ we have the following two cases: $a_n=a_{m+1}+b_{m+1}-a_m\neq 0$ or $a_{m+1}+b_{m+1}-a_m=0$. In the former case, since $a_n=a_{m+1}+b_{m+1}-a_m=a_{m+1}-a_{n-1}-b_{n-1}-a_m=-b_{n-1}-a_n$, we have $2a_n=-b_{n-1}$, which contradicts $-b_{n-1}=b_m=-a_m=-a_n$. And in the latter case, since $0=a_{m+1}+b_{m+1}-a_m=-b_{n-1}-a_m$, we have $a_m=-b_{n-1}$, which contradicts $a_m=-b_m=b_{n-1}$

$\neq 0$. Hence $a_m+b_m=0$ is impossible, as we asserted.

Thus from $a_n=a_m+b_m$ and $a_n=a_m$ we have $b_m=0$. By the skew symmetricity of $G(x)$ we have $a_{n-i}+b_{n-i}-a_{n-i-1}=-(a_{m+i+1}+b_{m+i+1}-a_{m+i})$ $(i=1,2,\cdots,n-m-1)$. On the other hand, we have $a_{n-i}=a_{m+i}$ Hence $b_{n-i}=-b_{m+1+i}$ $(i=0,1,\cdots,n-(m+1))$; thus the first part (I) of the conclusion of our Lemma results.

Case 4. $b_n=-b_m\neq 0$. Similar consideration as the case 3 leads to the latter half (II) of the conclusion of our Lemma.

By a simple calculation we have from Lemma 1 directly,

Lemma 2. *$f(x)$, $g(x)$, $F(x)$ and $G(x)$ having the same meaning as in* Lemma 1,

$$xf(x)-g(x)=x^p\{f(x^{-1})+g(x^{-1})\}$$

where p is a suitably chosen integer.

2. Now let κ' be a symmetric skew union of a given knot κ. We are going to consider the Alexander polynominal $\Delta_{\kappa'}(x)$ of κ'. We may suppose that the winding number is equal to 1.[2] Let the projection κ'_E of κ' on the ground plane E assume the form as shown in Fig. 1.

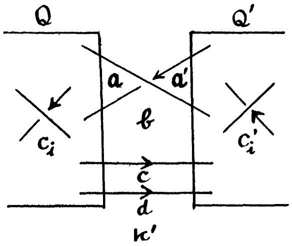

Fig. 1

We now introduce a new knot and a link κ_1 and κ_2 as defined in Fig. 2 and Fig. 3.

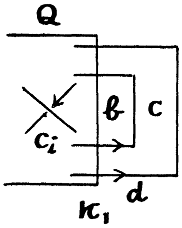

Fig. 2

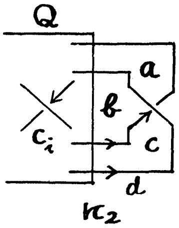

Fig. 3

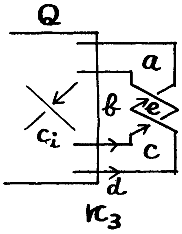

Fig. 4

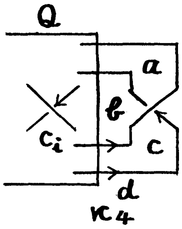

Fig. 5

It it clear that either i) κ_1 is a knot and κ_2 is a link of multiplicity 2, or ii) κ_2 is a knot and κ_1 is a link of multiplicity 2.

$\Delta_{\kappa_i}(x)$ or $\Delta_{\kappa_i}(x,x)$ denoting the Alexander polynomials corresponding to κ_i, in the case i) we put

$$f(x)=\pm x^{p_1}\Delta_{\kappa_1}(x) \quad \text{and} \quad g(x)=\pm x^{p_2}(x-1)\Delta_{\kappa_2}(x,x)\text{[3]}$$

and in the case ii),

$$f(x)=\pm x^{p_1}(x-1)\Delta_{\kappa_1}(x,x) \quad \text{and} \quad g(x)=\pm x^{p_2}\Delta_{\kappa_2}(x),$$

where p_1 and p_2 are suitably chosen integers.

Then we have

Theorem. *If κ' is a symmetric skew union of a knot κ, then the Alexander polynomial $\Delta_{\kappa'}(x)$ is of the following form;*

2) See Theorem 3 of [3].
3) See Theorem, Chap. I of [5].

$$\pm x^p \Delta_{\kappa'}(x) = \{f(x)+g(x)\}\{f(x^{-1})+g(x^{-1})\}$$

where p is a suitably chosen integer and $f(x)$ and $g(x)$ have the above meaning.

Proof. Since $\Delta_{\kappa'}(x)$ is independent of the choice of orientation of κ', we may suppose that κ' is oriented as in Fig. 1. Then the Alexander matrix M of κ' will take the following form;

$$M=\begin{array}{c} \begin{array}{ccccccc} b & c & a & c_1\cdots c_m & d & a' & c_1\cdots c_m \end{array} \\ \left(\begin{array}{c|c|c|c|c|c|c} * & * & \begin{matrix} a_1 \\ \vdots \\ a_{m+1} \end{matrix} & c_{ij} & \begin{matrix} d_1 \\ \vdots \\ d_{m+1} \end{matrix} & 0 & 0 \\ \hline x & 0 & -1 & 0 & 1 & -x & 0 \\ \hline * & * & 0 & 0 & \begin{matrix} -d_1 \\ \vdots \\ -d_{m+1} \end{matrix} & \begin{matrix} -a_1 \\ \vdots \\ -a_{m+1} \end{matrix} & -c_{ij} \end{array}\right) \end{array}$$

where $i=1,2,\cdots,m+1$ and $j=1,2,\cdots,m$.

To calculate the Alexander polynomial, first reduce M to a square matrix by striking out two columns corresponding to regions b and c. Then adding each $(m+2+i)$-th row $(i=1,2,\cdots,m+1)$ to the i-th row and then each j-th column $(j=1,2,\cdots,m+1)$ to the $(m+2+j)$-th column respectively, we have further

$$\left(\begin{array}{c|c|c|c|c} \begin{matrix} a_1 \\ \vdots \\ a_{m+1} \end{matrix} & c_{ij} & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & -1-x & 0 \\ \hline 0 & 0 & \begin{matrix} -d_1 \\ \vdots \\ -d_{m+1} \end{matrix} & \begin{matrix} -a_1 \\ \vdots \\ -a_{m+1} \end{matrix} & -c_{ij} \end{array}\right).$$

Since the matrices corresponding to κ_1 and κ_2 take the forms

$$M_1=\begin{array}{c} \begin{array}{cccc} b & c & d & c_1\cdots c_m \end{array} \\ \left(\begin{array}{c|c|c|c} * & * & \begin{matrix} d_1 \\ \vdots \\ d_{m+1} \end{matrix} & c_{ij} \end{array}\right) \end{array} \quad \text{and} \quad M_2=\begin{array}{c} \begin{array}{ccccc} b & c & d & a & c_1\cdots c_m \end{array} \\ \left(\begin{array}{c|c|c|c|c} -x & 1 & -1 & x & 0 \\ \hline * & * & \begin{matrix} d_1 \\ \vdots \\ d_{m+1} \end{matrix} & \begin{matrix} a_1 \\ \vdots \\ a_{m+1} \end{matrix} & c_{ij} \end{array}\right) \end{array},$$

we have

$$f(x)=\left|\begin{array}{c|c} \begin{matrix} d_1 \\ \vdots \\ d_{m+1} \end{matrix} & c_{ij} \end{array}\right| \quad \text{and} \quad g(x)=\left|\begin{array}{c|c} \begin{matrix} a_1 \\ \vdots \\ a_{m+1} \end{matrix} & c_{ij} \end{array}\right| + x\left|\begin{array}{c|c} \begin{matrix} d_1 \\ \vdots \\ d_{m+1} \end{matrix} & c_{ij} \end{array}\right|.$$

Therefore we have

$$\pm x^p \Delta_{\kappa'}(x) = \{f(x)+g(x)\}\{xf(x)-g(x)\}$$

where p is a suitably chosen integer.

Our proof will be complete if we show that $f(x)$ and $g(x)$ satisfy the conditions of Lemma 2.

For this purpose let us introduce a new knot and a link κ_3 and κ_4 as defined in Fig. 4 and Fig. 5.

It is clear that in the case i) κ_3 is a knot and κ_4 is a link of multiplicity 2, and in the case ii) κ_3 is a link of multiplicity 2 and κ_4 is a knot.

Moreover it is clear that it suffices to prove the theorem only for the case i).

Then the matrices M_3 and M_4 of κ_3 and κ_4 take the following forms;

$$M_3=\begin{array}{c} \begin{array}{cccccc} b & c & e & d & a & c_1\cdots c_m \end{array} \\ \left(\begin{array}{c|c|c|c|c|c} -x & 0 & 1 & -1 & x & 0 \\ \hline -x & 1 & x & -1 & 0 & 0 \\ \hline & & & d_1 & a_1 & \\ * & * & 0 & \vdots & \vdots & c_{ij} \\ & & & d_{m+1} & a_{m+1} & \end{array}\right) \end{array}, \quad M_4=\begin{array}{c} \begin{array}{ccccc} b & c & d & a & c_1\cdots c_m \end{array} \\ \left(\begin{array}{c|c|c|c|c} x & -x & 1 & -1 & 0 \\ \hline & & d_1 & a_1 & \\ * & * & \vdots & \vdots & c_{ij} \\ & & d_{m+1} & a_{m+1} & \end{array}\right) \end{array}.$$

We have therefore

$$\pm x^{p_3}\Delta_{\kappa_3}(x)=xf(x)+(x-1)g(x),$$
$$\pm x^{p_4}(x-1)\Delta_{\kappa_4}(x, x)=(1-x)f(x)+g(x)$$

where p_3 and p_4 are suitably chosen integers.

Since $\Delta_{\kappa_1}(x)$ and $\Delta_{\kappa_3}(x)$ are symmetric and of even degree by a theorem of Seifert [4] and since $\Delta_{\kappa_2}(x, x)$ and $\Delta_{\kappa_4}(x, x)$ are skew symmetric by a theorem of Torres (Theorem I, Chap. II of [5]), $f(x)$ and $g(x)$ are thus seen to satisfy the conditions of Lemma 2, and the proof of the theorem is complete.

Given a link of multiplicity 2, put it in the position κ_2 as Fig. 3. Then taking the link κ_2 and the knot κ_1 corresponding to κ_2 in Fig. 2 into account, we obtain by use of Lemma 1

Corollary. *If κ is a link of multiplicity* 2, *then the polynomial* $\Delta_\kappa(x, x)$ *of κ has an even degree.*

References

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[5] G. Torres: On the Alexander polynomial, Ann. Math., **57**, 57-89 (1953).