18. *Quasiideals in Semirings without Zero*

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O. Steinfeld [2, 3] has introduced the notion of quasiideals in rings, and semigroups and proved some interesting theorems. In this paper, we shall consider and prove some theorems on quasiideals in semirings. For fundamental concepts on a semiring and its related subjects, we shall follow the papers by S. Bourne [1], H. S. Vandiver and M. W. Weaver [4]. Unless otherwise stated, the word *semiring* shall mean *semiring without zero.*

Let S be a semiring, and suppose that A is a subset of S which is additively closed: if $a, b \in A$, then $a+b \in A$. A is a *quasiideal* if and only if $AS \frown SA \subset A$. Any quasiideal A is subsemiring of S, since $A^2 \subset AS \frown SA \subset A$. The intersection $\bigcap_\alpha A_\alpha$ of quasiideals A_α of S is empty or a quasiideal. For, if $A=\bigcap_\alpha A_\alpha \neq \phi$, then, for each α, $AS \frown SA \subset A_\alpha S \frown SA_\alpha \subset A_\alpha$, and we have $AS \frown SA \subset A$.

Lemma 1. The intersection of a right ideal and a left ideal in a semiring is a quasiideal.

Proof. Let R be a right ideal in S, and L a left ideal in S, then $RL \subset R \frown L$ and $R \frown L$ is not empty. Further, we have

$$(R \frown L)S \frown S(R \frown L) \subseteq RS \frown SL \subseteq R \frown L,$$

and this shows that $R \frown L$ is a quasiideal.

Lemma 2. Let ε be a multiplicative idempotent, and L a left ideal, R a right ideal in a semiring S, then εL and $R\varepsilon$ are quasiideal and

$$\varepsilon L = L \frown \varepsilon S, \quad R\varepsilon = S\varepsilon \frown R.$$

Proof. By Lemma 1, it is sufficient to prove the relations $\varepsilon L = L \frown \varepsilon S$ and $R\varepsilon = S\varepsilon \frown R$. As it is trivial that $\varepsilon L \subseteq L \frown \varepsilon S$, we shall show $\varepsilon L \supseteq L \frown \varepsilon S$. Let a be an element of $L \frown \varepsilon S$, then we have

$$a = \varepsilon S,$$

$s \in S$ and $a \in L$.
Hence, since $\varepsilon^2 = \varepsilon$, we have

$$\varepsilon a = \varepsilon \cdot \varepsilon S = \varepsilon S$$

and this shows $\varepsilon s = \varepsilon a \in \varepsilon L$ and we have $L \frown \varepsilon S \subset \varepsilon L$, similary, for right ideal R, we have $R\varepsilon = S\varepsilon \frown R$.

Theorem 1. The intersection of minimal right and minimal left ideals in a semiring is a minimal quasiideal.

Proof. Let R and L be minimal right and left ideals in the semiring S, and let Q be the intersection of R and L, then Q is a nonempty quasiideal by Lemma 1. Suppose that Q is not minimal, so there is a quasiideal Q' such that $Q' \subseteq Q$. Then we have $Q' \subseteq L$, and since L is minimal, $SQ'=L$. Similarly, we have $Q'S=R$. Hence $Q=L \frown R=SQ' \frown Q'S \subseteq Q'$, which contradicts.

Theorem 2. Every minimal quasiideal Q in a semiring S is represented as follows:

$$Q = Sa \frown aS,$$

where a is any element of Q, Sa is a minimal left ideal, and aS is a minimal right ideal.

Proof. For an element a of Q, by Lemma 1, $Sa \quad aS$ is a quasiideal in S, and we have

$$Sa \frown aS \subseteq SQ \frown QS \subseteq Q.$$

Since Q is a minimal quasiideal, $Q=Sa \frown aS$.

To prove that Sa is a minimal left ideal, suppose that L is a left ideal such that $L \subseteq Sa$, then we have

$$SL \subseteq L \subseteq Sa.$$

Therefore,

$$SL \frown aS \subseteq Sa \frown aS=Q.$$

By Lemma 1, $SL \frown aS$ is a quasiideal, and further, since Q is minimal, $SL \frown aS=Q$. On the other hand, by $Q \subseteq Sa \subseteq SL$, we have $Sa \subseteq SQ \subseteq SL$. This shows $L=Sa$, and it means that Sa is a minimal left ideal. Similarly, aS is a minimal right ideal. Therefore the proof is complete.

Let Q be a minimal quasiideal in a semiring S. By Theorem 2, for any element a of Q, we have

$$Sa \frown aS = Q,$$
$$Sa^2 \frown a^2S = Q.$$

Therefore, for an element b, there are four elements p, q, r and S in S such that

$$b = pa = aq,$$
$$b = ra^2 = a^2S.$$

Hence, we can find two elements x, y such that

$$a = xa^2 = a^2y,$$

and we have $xa^2y=xa=ay \in Sa \frown aS=Q$. Then $xaxa=xaay=xa$. This shows that xa is an idempotent in S. Let e be the idempotent, then $e \in Q$, and, by Theorem 2, we have a presentation of Q: $S\varepsilon \frown \varepsilon S=Q$. By Lemma 2, $\varepsilon S\varepsilon$ is a quasiideal and $\varepsilon S\varepsilon \subseteq Q$, therefore $\varepsilon S\varepsilon=Q$. The idempotent ε is the unit element of the subsemiring Q of S. We shall show that Q is a group on the multiplication. For an element $\varepsilon a\varepsilon$ of Q, we have $\varepsilon S\varepsilon \cdot \varepsilon a\varepsilon \subseteq \varepsilon S\varepsilon=Q$. By Lemma 2, $\varepsilon S\varepsilon \cdot \varepsilon a\varepsilon$ is a quasiideal in S, therefore we have $\varepsilon S\varepsilon \cdot \varepsilon a\varepsilon=\varepsilon S\varepsilon$. This shows that the equation $x\ \varepsilon a\varepsilon=\varepsilon b\varepsilon$ is solvable in $\varepsilon S\varepsilon$. Similarly $\varepsilon S\varepsilon \cdot x=\varepsilon b\varepsilon$ is solvable. Hence Q is a group on the multiplication, i.e. a division semiring in the sense of S. Bourne [1].

Conversely, suppose that a quasiideal Q in a semiring S is a division semiring, then Q is minimal. To prove it, let Q' be a quasiideal of S such that $Q' \subseteq Q$, then $Q'Q \frown QQ' \subseteq Q'S \frown SQ' \subseteq Q'$ and Q' is a quasiideal of Q. Let a be an element in Q', b an element in Q, then $ax=b$ and $ya=b$ are solvable in Q. Therefore $b \in aQ \frown Qa \subseteq Q'Q \frown QQ' \subseteq Q'$. This shows $Q=Q'$. Hence Q is minimal. Therefore we have the following fundamental

Theorem 3. If there is a minimal quasiideal Q in a semiring S:
(1) *There is at least one idempotent ε in Q.*
(2) $Q=\varepsilon S \varepsilon$.
(3) *Q is a division semiring.*

Corollary. A quasiideal in a semiring is minimal, if and only if it is a division semiring.

Theorem 4. Minimal quasiideals of a semiring are all isomorphic together.

Proof. Let Q_1, and Q_2 be two quasiideals in a semiring S, then $Q_1=\varepsilon_1 S \varepsilon_2$, $Q_2=\varepsilon_2 S \varepsilon_2$ by Theorem 3. Let a be an element of S, then $\varepsilon_1 a \varepsilon_1 \cdot \varepsilon_2 S \varepsilon_1 \subseteq \varepsilon_1 S \varepsilon_2 \varepsilon_2 S \varepsilon_1 \subseteq \varepsilon_1 S \varepsilon_1$, and $\varepsilon_1 a \varepsilon_2 \varepsilon_2 S \varepsilon_1 = \varepsilon_1 S \varepsilon_1$. Hence, there is an element b of S such that

$$\varepsilon_1 a \varepsilon_2 \varepsilon_2 b \varepsilon_1 = \varepsilon_1.$$

The element $\varepsilon_2 b \varepsilon_1 \varepsilon_1 a \varepsilon_2$ is idempotent of $\varepsilon_2 S \varepsilon_2$, for $(\varepsilon_2 b \varepsilon_1 \cdot \varepsilon_1 a \varepsilon_2)^2 = \varepsilon_2 b \varepsilon_1 \varepsilon_1 a \varepsilon_2 \times \varepsilon_2 b \varepsilon_1 \varepsilon_1 a \varepsilon_2 = \varepsilon_2 b \varepsilon_1 \varepsilon_1 a \varepsilon_2 \in \varepsilon_2 S \varepsilon_2$. Therefore $\varepsilon_1 x \varepsilon_1 \rightarrow \varepsilon_2 b \varepsilon_1 \varepsilon_1 x \varepsilon_1 \varepsilon_1 a \varepsilon_2$ is a mapping φ from Q_1 to Q_2. Since Q_1, Q_2 are division semirings, the mapping is one-to-one. If x and y are elements of Q_1, we have $\varphi(x+y)=\varphi(x)+\varphi(y)$. For x and y of Q_1, since $X\varepsilon_1 \cdot \varepsilon_1 y \in Q$ and $\varepsilon_1 a \varepsilon_2 \varepsilon_2 b \varepsilon_1 = \varepsilon_1$, we have $\varepsilon_1 x \varepsilon_1 \varepsilon_1 y \varepsilon_1 \rightarrow \varepsilon_2 b \varepsilon_1 x \varepsilon_1 \varepsilon_1 y \varepsilon_1 a \varepsilon_2 = \varepsilon_2 b \varepsilon_1 \varepsilon_1 x \varepsilon_1 \cdot \varepsilon_1 a \varepsilon_2 \varepsilon_3 b \varepsilon_1 \varepsilon_1 y \varepsilon_1 \cdot \varepsilon_1 a \varepsilon_2$, and this shows $\varphi(xy)=\varphi(x)\varphi(y)$. Hence φ is homomorphism and Q_1 and Q_2 are isomorphic, the proof is complete.

References

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