17. *On Some Function Spaces concerning Dirichlet's Problem*

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§ 1. Let E be the n-dimensional Euclidean space for a certain n ($\geqq 3$), and denoting the Euclidean distance in E by $r(x, y)$ we define the Newtonian potential

$$\phi(\mu)(x)=N_n\int r^{2-n}(x, y)\, d\mu(y), \quad N_n=\frac{\Gamma(n/2)}{2(n-1)\pi^{n/2}}$$

for any positive Radon measure μ in E.

Let D be a given domain in E, whose closure $\overline{D}$ and hence boundary ∂D are both compact. For each positive measure μ distributed in $\overline{D}$, consider the *inner balayage* μ_{Γ}^{0} of μ in ∂D and the *outer balayage* μ_{∇}^{0} of μ in $\partial\overline{D}$ (about these matters, see my another paper "On the foundation of balayage theory" which will appear in the Journal of Polytech., Osaka City Univ., 9, no. 2; cited hereafter as [1]). The notations and results used here shall be referred to that paper [1], but some of those are quoted for convenience' sake as follows.

For a measurable set X in E, we define

$C(X)$ (or $C_u(X)$)=space of all bounded (resp. uniformly) continuous functions defined in X,

$\mathfrak{M}^+(X)$ (or $\mathfrak{M}_0^+(X)$)=collection of all positive Radon measures distributed in X (resp. of norm less that 1). $\mathfrak{M}(X)$=linear envelope of $\mathfrak{M}^+(X)$ on the reals.

Γ_0=set of all inner regular boundary-points of D (*i.e.* $(\varepsilon_x)_{\Gamma}^{0}=\varepsilon_x$ whenever $x\in\Gamma_0$).

∇_0=set of all outer regular (or in other words, *stable*) boundary-points of $\overline{D}$ (*i.e.* $(\varepsilon_x)_{\nabla}^{0}=\varepsilon_x$ whenever $x\in\nabla_0$).[1]

$H(D)$=normed linear space consisting of the restrictions in $\overline{D}$ of all bounded potentials $f=\phi(\mu)$ for $\mu\in\mathfrak{M}(E-D)$ with respect to the norm

(1.1) $$\|f\|_D=\sup_{x\in\overline{D}}|f(x)|.$$

We see that $\nabla_0\subset\Gamma_0\subset\partial D$ and $\partial D-\Gamma_0$ is of (inner) capacity 0.

§ 2. We now define a linear normed space $\Phi(\Gamma_0)$ linearly generated on Γ_0 from the collection of all potentials $\phi_x=\phi(\varepsilon_x)$ for $x\in E-\overline{D}$ with respect to the norm

(2.1) $$\|\phi_x\|_{\Gamma_0}=\sup_{y\in\Gamma_0}|\phi_x(y)|.$$

1) ε_x designates a point measure of total mass $+1$ placed on $x\in E$.

We have seen in [1]:

Theorem A. *The normed linear space $H(\Gamma_0)$ of restrictions in Γ_0 of all functions of $H(D)\frown C_u(\Gamma_0)$ with respect to the uniform norm on Γ_0 is dense in the Banach space $C_u(\Gamma_0)$* (Theorem 19 [1]).

This theorem conducts us to a new solution of Dirichlet's problem as a Banach space method (as is mentioned in [1]).

We remark now that this involves the noted theorem of M. Keldych [2] and its extension given by N. Ninomiya [3], since every function of $H(D)\frown C_u(\Gamma_0)$ has evidently a solution of the classical Dirichlet's problem.

Our main results in the present paper is:

Theorem B. *Let μ be any positive measure in $\mathfrak{M}^+(\partial D)$ with bounded $\phi(\mu)$; the positive linear functional $\mu^\wedge$ on $\Phi(\Gamma_0)$ defined by*

$$\mu^\wedge(f)=\int_{\Gamma_0} f\,d\mu \quad \text{for } f\in\Phi(\Gamma_0) \tag{2.2}$$

is uniquely prolonged up to a linear positive functional on $C_u(\Gamma_0)$ if and only if D is stable.[2]

Here, we say that D is stable if for every $f\in C(\partial D)$ we have $\tilde{f}(x)=f^*(x)$ everywhere in D, where $\tilde{f}(x)=\int_{\partial D} f\,d(\varepsilon_x)^0_\Gamma$ and $f^*(x)=\int_{\partial D} f\,d(\varepsilon_x)^0_\nabla$: this is equivalent to that $(\varepsilon_x)^0_\Gamma=(\varepsilon_x)^0_\nabla$ for any $x\in D$ or (owing to M. Brelot) $\partial D-\nabla_0$ is of capacity 0.

Proof of Theorem B. Suppose first that $\partial D-\nabla_0$ is of capacity 0 but yet there were another extension $\xi^\sim$ of $\mu^\wedge$ on $C_u(\Gamma_0)$ than that defined by

$$\mu^\sim(f)=\int_{\Gamma_0} f\,d\mu \quad \text{for } f\in C_u(\Gamma_0).\text{[3]} \tag{2.2$'$}$$

Now, $\xi^\sim$ defines a positive Radon measure ξ on $\overline{\Gamma_0}$ and, since $\xi^\sim=\mu^\sim$ on $\Phi(\Gamma_0)$, we see immediately that $\xi^0_\nabla=\mu^0_\nabla=\mu$ (μ can not be distributed outside of ∇_0 as $\partial D-\nabla_0$ is of capacity 0 by hypothesis). On the other hand, $\phi(\xi)(y)=\phi(\xi^0_\nabla)(y)=\phi(\mu)(y)$ for all $y\in E-\overline{D}$ and, since $\phi(\mu)$ is bounded ($\leqq K<+\infty$), we have $\phi(\xi)(y)\leqq K$ for $y\in E-\overline{D}$ and $\phi(\xi)(x)\leqq\varliminf_{y\to x}\phi(\xi)(y)\leqq K$ for every $x\in\partial\overline{D}$ and $y\in E-\overline{D}$ such that $y\to x$; thus, $\phi(\xi)$ is bounded on $\partial\overline{D}$ and hence on the support of ξ; this implies by maximum principle that $\phi(\xi)$ is bounded everywhere in E and consequently ξ must be distributed on ∇_0. This concludes that $\xi=\xi^0_\nabla=\mu$, contradicting with the assumption that $\xi^\sim\neq\mu^\sim$.

Conversely, if D is not stable, it holds $(\varepsilon_x)^0_\Gamma\neq(\varepsilon_x)^0_\nabla$ for a certain $x\in D$, nevertheless $(\varepsilon_x)^{0\wedge}_\Gamma=(\xi_x)^{0\wedge}_\nabla$ on $\Phi(\Gamma_0)$. This completes the proof of Theorem B.

2) 3) Since $\phi(\mu)$ is bounded and $\partial D-\Gamma_0$ is of capacity 0, μ is distributed in Γ_0.

Theorem B′. *The functional* $\varepsilon_x^{\wedge}(\phi_z)=\phi_z(x)$ *on* $\Phi(\Gamma_0)$ *for each* $x\in D$ *is uniquely prolonged up to a positive linear functional on* $C_u(\Gamma_0)$ *if and only if* D *is stable.*

In fact, $\varepsilon_x^{\wedge}(\phi_z)=\phi_z(x)=\int_{\partial D}\phi_z\, d(\varepsilon_x)_\Gamma^0$ and $\phi((\varepsilon_x)_\Gamma^0)$ is bounded, therefore the proof of the preceding Theorem is entirely valid in this case.

We see that Theorem B′ yields a criterion of the stability of domain relating to the functional determination of the solution of Dirichlet's problem, early obtained by M. Inoue [4] in a different way, that is:

Theorem C (M. Inoue). *Let* $\varepsilon_x^{\wedge}$ *be a positive linear functional related to a point* $x\in D$ *on* $C(\partial D)$, *satisfying the condition that for each* ϕ_z, $z\in E-\overline{D}$,

$$\varepsilon_x^{\wedge}(\phi_z)=N_n\cdot r^{2-n}(x, Z), \tag{2.3}$$

then $\varepsilon_x^{\wedge}(f)=\tilde{f}(x)$ (solution of Dirichlet's problem) *for every* $f\in C(\partial D)$, *if and only if* D *is stable.*

In fact, to introduce Theorem C from Theorem B′ it is sufficient to remark that $C(\partial D)$ forms a linear subspace of $C_u(\Gamma_0)$ and conversely every function of $C_u(\Gamma_0)$ is well prolonged over the whole ∂D.

References

[1] S. Matsushita: On the foundation of balayage theory, Jour. Polytech., Osaka City Univ., **9**, no. 2 (1958) (to be issued).

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[4] M. Inoue: Sur la détermination fonctionnelle de la solution du problème de Dirichlet, Mem. Fac. Sci. Kyushu Univ., **5**, 69–74 (1950).