16. *One Point Expansion of Topological Spaces*

By Eiichi HAYASHI
Aichi Gakugei University

(Comm. by K. KUNUGI, M.J.A., Feb. 12, 1958)

It is well known as the Alexandroff one point compactification that any topological space can be compactified by adjoining a single point (for instance [1, 149-150]). In this note, we begin by defining the one point expansion without requiring compactness (Definition 2) and proving that any topological space has the one expansion (Theorem 1). We further obtain that the Alexandroff one point compactification is a special case of our one point expansion, and conclude this paper by showing another example of our expansion (Theorem 2).

DEFINITION 1. Let $\mathfrak{M}$ be a collection of subsets of a topological space S and suppose that $\mathfrak{M}$ satisfies the following two conditions:
(1) Heredity. If a subset $A\subset S$ belongs to $\mathfrak{M}$ and $B\subset A$, then B belongs to $\mathfrak{M}$.
(2) Finitely additivity. If A and $B\subset S$ belong to $\mathfrak{M}$, then $A\smile B$ belongs to $\mathfrak{M}$.

For example, the collections of the empty set, of finite sets, of nowhere dense sets and of meager sets, respectively, of a topological space satisfy those two conditions.

We shall first prove

LEMMA 1. *Let R be a topological space which does not belong to $\mathfrak{M}$ and p a new point. Then we can construct an enlarged space $R^*=R\smile p$, having the topology decided by the collection $\mathfrak{M}$ and possessing the following properties:*

(1) R^* is a topological space.

(2) R is a dense subspace of R^*.

(3) If R is a T_0-space, then R^* is a T_0-space. Further if R is dense in itself, then R^* is dense in itself.

PROOF. For any $X\subset R^*=R\smile p$, we use $\overline{X}$ and $\tilde{X}$ to denote the closure of a set X in the initial topology and the new topology, respectively. We define the topology in R^*, as follows:

$$\tilde{X}=\begin{cases}\overline{X} & X\subset R, \text{ if } \overline{X} \text{ belongs to } \mathfrak{M}.\\ \overline{X}\smile p & X\subset R, \text{ if } \overline{X} \text{ does not belong to } \mathfrak{M}.\\ \overline{X-p}\smile p, & \text{if } p \text{ belongs to } X.\end{cases}$$

(1) We omit the proof, which consists of tiresome verifications.

(2) Since R does not belong to $\mathfrak{M}$, we have

$$\tilde{R}=\overline{R}\smile p=R^*.$$

It is easily seen that, for any $X \subset R$, $R \frown \widetilde{X} = \overline{X}$.

(3) Assume that R is a T_0-space. Then for any two points a, $b \in R$, there is a subset A of R such that $a \in \overline{A}$ and $b \bar{\in} \overline{A}$. If $\overline{A}$ belongs to $\mathfrak{M}$, then our conclusion is obvious. If $\overline{A}$ does not belong to $\mathfrak{M}$, then $a \in \overline{A} \subset \widetilde{A}$ and $b \bar{\in} \overline{A} \smile p = \widetilde{A}$. On the other hand, for any point $a \in R$ and a point p, we have that $p \in \widetilde{p}$ and $a \neq p = \widetilde{p}$. Thus R^* is a T_0-space.

Next, assume that R is dense in itself, and denote the derived set and the isolated set of R in the topology of R by $d(R)$ and $s(R)$, respectively. Then for any $q \in R$, we have

$$q \in \overline{R} = d(R) \smile s(R) \text{ and } q \bar{\in} s(R).$$

Hence
$$q \in \overline{R-q} \subset \widetilde{R-q},$$

and since p is an accumulation point of R^*, it follows that R^* is dense in itself, which completes the proof.

In respect of Lemma 1, we make the following

DEFINITION 2. The topological space R^* in Lemma 1 is called the one point expansion of R.

In Lemma 1, if $\mathfrak{M}$ means the collection of empty sets, we have directly the following

THEOREM 1. Any topological space has the one point expansion.

We shall now impose a stronger condition on the collection.

LEMMA 2. Suppose that the collection $\mathfrak{M}$ satisfies the next condition (3), in addition to the conditions (1) and (2) in Definition 1. Then, for any neighbourhood $N(p)$ of p, $R^* - N(p)$ belongs to $\mathfrak{M}$.

(3) If $A \subset S$ belongs to $\mathfrak{M}$ in the topology of S, then A belongs to $\mathfrak{M}$ in the coarser topology.

PROOF. Since $p \bar{\in} \widetilde{R^* - N(p)}$, $R^* - N(p)$ belongs to $\mathfrak{M}$ in the topology of both R and R^*, by the definition of the new closure. q.e.d.

In order to prove the Alexandroff theorem, we apply Lemmas 1 and 2.

THEOREM (Alexandroff). Any topological space R has the one point compactification R^*. Moreover, if R is a T_1-space then R^* is also a T_1-space.

PROOF. In our case, we take $\mathfrak{M}$ as a collection of subsets whose closures are compact. Then we have the one point expansion R^*. Since the topology in R^* is coarser than the one in R, we may consider that the collection $\mathfrak{M}$ satisfies the condition (3). Therefore, using the notation of Lemma 2, $R^* - N(p)$ is compact, and so there exists a finite open covering $\{G_i\}$ $(i=1,\cdots,n)$ of $R^* - N(p)$ such that

$$R^* - N(p) = G_1 \smile G_2 \smile \cdots \smile G_n.$$

Then, there exist open sets U_i $(i=1,\cdots,n)$ in R^* such that

$$U_i \frown (R^* - N(p)) = G_i.$$

Hence $$R^* \subset U_1 \smile U_2 \smile \cdots \smile U_n \smile N(p).$$

The remainder of this theorem can be readily shown. q.e.d.

The next theorem asserts that, under certain limitations, a non-meager space can be embedded in a meager space.

THEOREM 2. Suppose that R is a non-meager T_1-space and dense in itself. Then, the one point expansion R^* of R is a T_1-space. Moreover if there are at most countable neighbourhoods $N_i(p)$ of p, then R is meager in R^*.

PROOF. First, we remark that the collection of meager subsets of R satisfies the conditions (1), (2) and (3). From the fact that R is dense in itself, any point $a \in R$ is nowhere dense in R. Hence we have $\tilde{a} = \bar{a} = a$, and R^* is therefore a T_1-space.

By our assumption and T_1-axiom, we have $p = \bigcap_{i=1}^{\infty} N_i(p)$. Then, because every set $R^* - N_i(p)$ is meager in R^*, their countable sum

$$\bigcup_{i=1}^{\infty} (R^* - N_i(p)) = R^* - p = R$$

is meager in the one point expansion R^*.

Reference

[1] J. L. Kelley: General Topology (1955).