35. On Countable-Dimensional Spaces

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As is well known, not every infinite-dimensional metric space is the countable sum of zero-dimensional spaces; in fact the Hilbert-cube I_ω is not the countable sum of 0-dimensional spaces. It is known that by the generalized decomposition-theorem due to M. Katětov [1] and to K. Morita [2] a metric space is the countable sum of 0-dimensional spaces if and only if it is the countable sum of finite-dimensional spaces. We call such a space a *countable-dimensional space*. It seems, however, that our knowledge of countable-dimensional spaces is, because of peculiar difficulties to the infinite-dimensional case, very little if compared to that of finite-dimensional spaces.

The purpose of this note is to extend the theory of finite-dimensional spaces to the countable-dimensional case.[1)]

All spaces considered in the present note will be assumed to be metric spaces unless the contrary is explicitly stated. Dim R denotes the Lebesgue dimension of R.

We denote by $\text{order}_p\,\mathfrak{U}$ for a point p and for a covering $\mathfrak{U}$ of a space R the largest integer n such that there exist n members of $\mathfrak{U}$ which contain p. We also use the notation $B(\mathfrak{U})=\{B(U)\mid U\in\mathfrak{U}\}$, where $B(U)$ means the boundary of U.

Lemma 1. *Let A_n, $n=1,2,\cdots$ be a countable number of 0-dimensional sets of a space R. Let $\{U_\alpha\mid\alpha<\tau\}$[2)] be a collection of open sets and $\{F_\alpha\mid\alpha<\tau\}$ a collection of closed sets such that $F_\alpha\subset U_\alpha$, $\alpha<\tau$ and such that $\{U_\beta\mid\beta<\alpha\}$ is locally finite for every $\alpha<\tau$. Then there exists a collection of open sets V_α, $\alpha<\tau$ such that*

1) $F_\alpha\subset V_\alpha\subset U_\alpha$, $\alpha<\tau$,

2) $\text{order}_p\,B(\mathfrak{V})\leqq n-1$ *for every* $p\in A_n$,

where $\mathfrak{V}=\{V_\alpha\mid\alpha<\tau\}$.

Proof. We shall define, by induction with respect to α, satisfying 1) and

$2)_\alpha$ $\text{order}_p\,B(\mathfrak{V}_\alpha)\leqq n-1$ for every $p\in A_n$, where $\mathfrak{V}_\alpha=\{V_\beta\mid\beta\leqq\alpha\}$.
We take open sets G_1, W_1 such that

$$G_1\supset F_1,\quad W_1\supset U_1^c,\quad \overline{G}_1\frown\overline{W}_1=\phi.$$

Since A_1 is 0-dimensional, there exists an open, closed set N_1 of A_1 satisfying $\overline{G}_1\frown A_1\subset N_1\subset(\overline{W}_1)^c\frown A_1$. If we put $B_1=N_1\smile F_1$, $C_1=(A_1-N_1)$

1) The detail of the content of this note will be published in an another place.
2) We denote by α,β,γ,τ ordinal numbers.

$\smile U_1^c$, then $(\overline{B}_1 \frown C_1) \smile (B_1 \frown \overline{C}_1) = \phi$. Hence there exists an open set V_1 such that $B_1 \subset V_1 \subset \overline{V}_1 \subset C_1^c$. Since $B(V_1) \frown A_1 = \phi$ is clear, V_1 satisfies 1) and $2)_\alpha$ for $\alpha = 1$.

Suppose that V_β has been constructed for every $\beta < \alpha$ $(< \tau)$. Then we put

$H_1 = A_1$,

$H_n = \smile \{B(V_{\beta_1}) \frown \cdots \frown B(V_{\beta_{n-1}}) \frown A_n \mid \beta_1, \cdots, \beta_{n-1} < \alpha\}$, $n = 2, 3, \cdots$

and put

$$K_\alpha = \bigcup_{n=1}^{\infty} H_n.$$

It follows from $\dim A_n = 0$, $n = 1, 2, \cdots$ that

$$\dim H_n \leqq 0, \quad n = 1, 2, \cdots .$$

We see easily that for every n $\bigcup_{i=1}^{n} H_i$ is open in K_α. Hence for every n $H_n - \bigcup_{i=1}^{n-1} H_i$ is a 0-dimensional F_σ-set. Hence we have, by the generalized sum-theorem [2], $\dim K_\alpha \leqq 0$. Consequently we can define, in the same way as for the case of $\alpha = 1$, an open set V_α such that

$$F_\alpha \subset V_\alpha \subset U_\alpha, \quad B(V_\alpha) \frown K_\alpha = \phi,$$

which implies $2)_\alpha$. This completes the proof.

We can easily prove the following two theorems and one lemma as the consequences of this lemma.

Theorem 1. *A space R is countable-dimensional if and only if*[3] *there exists a countable collection of locally finite open coverings $\mathfrak{V}_i$ such that $\mathfrak{V} = \bigcup_{i=1}^{\infty} \mathfrak{V}_i$ is a basis of open sets of R and* $\text{order}_p B(\mathfrak{V}) < +\infty$ *for every point p of R.*

Theorem 2. *A space R is countable-dimensional if and only if for every collections $\{U_\alpha \mid \alpha < \tau\}$ of open sets and $\{F_\alpha \mid \alpha < \tau\}$ of closed sets such that $F_\alpha \subset U_\alpha$, $\alpha < \tau$ and such that $\{U_\beta \mid \beta < \alpha\}$ is locally finite for every $\alpha < \tau$, there exists a collection of open sets V_α, $\alpha < \tau$ satisfying*

1) $F_\alpha \subset V_\alpha \subset U_\alpha, \quad \alpha < \tau,$

2) $\text{order}_p B(\mathfrak{V}) < +\infty$ *for every* $p \in R$,

where $\mathfrak{V} = \{V_\alpha \mid \alpha < \tau\}$.

Lemma 2. *Let A_n, $n = 1, 2, \cdots$ be a countable number of 0-dimensional sets of a space R. Let $\mathfrak{U} = \{U_\alpha \mid \alpha < \tau\}$ be a locally finite open covering. Then there exists a closed covering $\mathfrak{F} = \{F_\alpha \mid \alpha < \tau\}$ such that $\mathfrak{F} < \mathfrak{U}$ and* $\text{order}_p \mathfrak{F} \leqq n$ *for every $p \in A_n$.*

Theorem 3. *A space R is countable-dimensional if and only if there exists a countable collection of locally finite closed coverings of R satisfying*

1) *for every nbd (=neighborhood) $U(p)$ of every point p of R*

3) To prove the "only if" part of this theorem we use a theorem of A. H. Stone [4].

there exists some i with $S(p, \mathfrak{F}_i)\subset U(p)$,

2) $\mathfrak{F}_i=\{F(\alpha_1,\cdots,\alpha_i)\mid \alpha_k\in\Omega,\ k=1,\cdots,i\}$, *where $F(\alpha_1,\cdots,\alpha_i)$ may be empty,*

3) $F(\alpha_1,\cdots,\alpha_{i-1})=\cup\{F(\alpha_1,\cdots,\alpha_{i-1},\beta)\mid\beta\in\Omega\}$,

4) $\sup\{\text{order}_p\,\mathfrak{F}_i\mid i=1,2,\cdots\}<+\infty$ *for each point p of R.*

Proof. Let R be a countable-dimensional space with $R=\bigcup_{n=1}^{\infty}A_n$ for 0-dimensional A_n and $\mathfrak{S}_1>\mathfrak{S}_2>\cdots$ a uniformity of R. Then we shall define $\mathfrak{F}_i$ satisfying 2), 3), $\mathfrak{F}_i<\mathfrak{S}_i$ and $\text{order}_p\,\mathfrak{F}_i\leqq n$ for each point $p\in A_n$. We can define $\mathfrak{F}_1$ by Lemma 2. Assume that we have defined $\mathfrak{F}_k$ for every $k<i$; then we put $\mathfrak{F}_{i-1}=\{F_\alpha\mid\alpha<\tau\}$ for brevity. To obtain $\mathfrak{F}_i$ we shall define closed sets $F_{\alpha\beta}$, $\alpha<\tau$, $\beta\in\Omega$ such that

i) $F_\alpha=\cup\{F_{\alpha\beta}\mid\beta\in\Omega\}$, $\{F_{\alpha\beta}\mid\beta\in\Omega\}<\mathfrak{S}_i$,

ii) $\mathfrak{G}_\alpha=\{F_{\alpha'\beta}\mid\alpha'\leqq\alpha,\beta\in\Omega\}\cup\{F_{\alpha'}\mid\alpha'>\alpha\}$ is locally finite for every $\alpha<\tau$,

iii) $\text{order}_p\,\mathfrak{G}_\alpha\leqq n$ for every $\alpha<\tau$ and for each point $p\in A_n$.

First we define $F_{1\beta}$, $\beta\in\Omega$ as follows:

We let

$H_{r+s\ s}=\{p\mid \text{order}_p\{F_\alpha\mid 1<\alpha<\tau\}=s,\ p\in F_1\cap A_{r+s}\}$, $r=1,2,\cdots$, $s=0,1,\cdots$,

$$K_r=\bigcup_{s=0}^{\infty}H_{r+s\ s},\quad r=1,2,\cdots\ .$$

Since we can easily see that $\bigcup_{s'=1}^{s}H_{r+s'\ s'}$ is open in K_r and that $\dim H_{r+s\ s}\leqq 0$, we have $\dim K_r\leqq 0$, $r=1,2,\cdots$. Therefore we can define by Lemma 2 a locally finite closed covering $\mathfrak{G}_1'=\{F_{1\beta}\mid\beta\in\Omega\}$ of F_1 such that $\mathfrak{G}_1'<\mathfrak{S}_i$, $\text{order}_p\,\mathfrak{G}_1'\leqq n$ for every $p\in K_n$. It is easy to see that $\text{order}_p\,\mathfrak{G}_1\leqq n$ for $p\in A_n$ and for $\mathfrak{G}_1=\mathfrak{G}_1'\cup\{F_{\alpha'}\mid\alpha'>1\}$.

The method of defining $F_{\alpha\beta}$ is quite parallel with that of $F_{1\beta}$ except that we use F_α and $\{F_{\alpha'\beta}\mid\alpha'<\alpha,\beta\in\Omega\}\cup\{F_{\alpha'}\mid\alpha'>\alpha\}$ instead of F_1 and $\{F_{\alpha'}\mid\alpha'>1\}$, respectively. The "if" part of this theorem follows from [3].

The following is the direct consequence of this theorem.

Theorem 4. *A space R is countable-dimensional if and only if there exist a subset S of $N(\Omega)$ for suitable Ω and a closed continuous mapping f of S onto R such that for each point p of R the inverse image $f^{-1}(p)$ consists of finitely many points, where $N(\Omega)$ denotes the generalized Baire's 0-dimensional space.*[4]

Lemma 3. *Let R be a countable-dimensional space with $R=\bigcup_{n=1}^{\infty}A_n$,* $\dim A_n=0$. *Let $\{U_m\mid m=1,2,\cdots\}$ be a collection of open sets and $\{F_m\mid m=1,2,\cdots\}$ a collection of closed sets such that $F_m\subset U_m$, $m=1$,*

4) $N(\Omega)=\{(\alpha_1,\alpha_2,\cdots)\mid\alpha_i\in\Omega,\ i=1,2,\cdots\}$. We define the metric ρ of $N(\Omega)$ as follows: if $\alpha=(\alpha_1,\alpha_2,\cdots)$, $\beta=(\beta_1,\beta_2,\cdots)$, $\alpha_i=\beta_i$ for $i<n$, $\alpha_n\neq\beta_n$, then $\rho(\alpha,\beta)=1/n$. This notion is due to [2].

$2, \cdots$. *Then there exists a collection of open sets* U_{mr}, $m=1, 2, \cdots$, $|r|<\sqrt{2}/2m$, *r: rational such that*

1) $F_m \subset U_{mr} \subset \overline{U}_{mr} \subset U_{mr'} \subset \overline{U}_{mr'} \subset U_m$ *for* $r<r'$,

2) $\overline{U}_{mr} = \frown\{U_{mr'} \mid r'>r\}$, $U_{mr} = \smile\{\overline{U}_{mr'} \mid r'<r\}$,

3) $\text{order}_p \{B(U_{mr}) \mid m=1, 2, \cdots, |r|<\sqrt{2}/2m, r: \textit{rational}\} \leqq n-1$ *for each point* $p \in A_n$.

Proof. Let us sketch the process of defining U_{mr}. First we number all rational numbers with $|r|<\sqrt{2}/2m$ so that

$$r_{m1},\ r_{m2}<r_{m1}<r_{m3},\ r_{m4}<r_{m2}<r_{m5}<r_{m1}<r_{m6}<r_{m3}<r_{m7}, \cdots .$$

Then we put

$$N_{m1}=\{r_{m1}\},\ N_{m2}=\{r_{m2}, r_{m3}\},\ N_{m3}=\{r_{m4}, r_{m5}, r_{m6}, r_{m7}\}, \cdots .$$

We shall define U_{mr} satisfying 1), 3) and

2)$'$ if r_{mi} and r_{mk} are adjoining numbers contained in $\overset{s-1}{\underset{h=1}{\smile}} N_{mh}$, $r_{mj} \in N_{ms}$ and $r_{mi}<r_{mj}<r_{mk}$, then

$$U_{r_{mj}} \subset S_{1/s}(\overline{U}_{r_{mi}}) \quad \text{if } s \text{ is odd},$$

$$(\overline{U}_{r_{mj}})^c \subset S_{1/s}((\overline{U}_{r_{mk}})^c) \quad \text{if } s \text{ is even},$$

where we denote, for brevity, $U_{mr_{mi}}$ by $U_{r_{mi}}$, and use the notation $S_\varepsilon(U) = \{x \mid \inf\{\rho(x, y) \mid y \in U\}<\varepsilon\}$ for the distance $\rho(x, y)$ between x and y. We define, by induction, all U_{mr} in such an order that

$$U_{r_{11}},\ U_{r_{12}},\ U_{r_{21}},\ U_{r_{13}},\ U_{r_{22}},\ U_{r_{31}}, \cdots .$$

The method of defining U_{mr} is analogous to that of V_α in Lemma 1.

Definition. Let $\{\mathfrak{N}_i \mid i=1, 2, \cdots\}$ be a collection of star-finite open coverings of a space R. If $\mathfrak{N} = \overset{\infty}{\underset{i=1}{\smile}} \mathfrak{N}_i$ is a basis of open sets, then we call $\{\mathfrak{N}_i \mid i=1, 2, \cdots\}$ a *σ-star-finite basis*.

The following theorem is a direct consequence of Lemma 3.

Theorem 5. *Let R be a space with a σ-star-finite basis. Then R is countable-dimensional if and only if R is homeomorphic to a subset of $N(\Omega) \times R_\omega$ for suitable Ω, where we denote by R_ω the set of points in I_ω at most finitely many of whose coordinates are rational, and denote by $N(\Omega)$ the generalized Baire's 0-dimensional space for Ω.*

Corollary. *A separable space R is countable-dimensional if and only if it is homeomorphic to a subset of R_ω.*

References

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