32. *Measures in the Ranked Spaces*

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This paper is an attempt to introduce the notion of measure in ranked spaces[1] which is an extension of the measure in the sense of Lebesgue.[2] This will be the first step to the general measure theory in the ranked spaces.

In Section 1, as the preparation for Section 2, we study some properties of outer measures in topological spaces. In Section 2 we give outer measures in ranked spaces and study their properties. Some examples will be given in Section 3.

1. Let R be a space whose topology is given by a system of neighbourhoods which satisfies F. Hausdorff's axioms (A), (B) and (C).[3]

Definition 1.[4] A set function Γ, defined on the family of all subsets of R, is called an outer measure in R if the following conditions (1.1)-(1.4) are satisfied:

(1.1) $0\leq\Gamma(A)\leq+\infty$ for any subset A of R.

(1.2) $\Gamma(0)$[5]$=0$.

(1.3) $\Gamma(A)\leq\Gamma(B)$ whenever $A\subseteq B$.

(1.4) $\Gamma(\bigcup_{n=1}^{\infty}A_n)\leq\sum_{n=1}^{\infty}\Gamma(A_n)$ for every countable sequence $\{A_n\}$ of subsets of R.

Definition 2.[4] Let Γ be an outer measure in R. A subset A of R is Γ-measurable if, for every subset E of R, we have

(1.5) $$\Gamma(E)=\Gamma(E\frown A)+\Gamma(E\frown(R-A)).^{6)}$$

Theorem 1. *Let Γ be an outer measure in R which satisfies the following conditions* (1.6)-(1.9):

(1.6) *For every disjoint finite or countable family $\{v_n(p_n);\ n=1,2,\cdots\}$ of neighbourhoods, $\Gamma(\overline{\bigcup_n v_n(p_n)}^{\,7)}-\bigcup_n v_n(p_n))=0$ if $\Gamma(\bigcup_n v_n(p_n))<+\infty$.*

1) The notion of ranked spaces was introduced by Prof. K. Kunugi in the notes "Sur les espaces complets et régulièrement complets. I-III", Proc. Japan Acad., **30**, 553-556, 912-916 (1954); **31**, 49-53 (1955).

2) F. Hausdorff and A. Appert studied this problem in the metric spaces. S. Saks: Theory of the Integral (1937). A. Appert: Mesures normales dans les espaces distanciés, Bull. Sci. Math., **60**, 329-352, 368-380 (1936).

3) F. Hausdorff: Grundzüge der Mengenlehre, 213 (1914).

4) Cf. P. Halmos: Measure Theory (1954).

5) 0 denotes the empty set.

6) For two subsets A and B of R, $A-B$ denotes the set of all elements p such that $p\in A$ and $p\notin B$.

7) For any subset A of R, $\bar{A}$ denotes the closure of A.

(1.7) *For every disjoint finite family* $\{v_n(p_n);\ 1\leq n\leq N\}$ *of neighbourhoods,* $\Gamma(\bigcup_{n=1}^{N} v_n(p_n))=\sum_{n=1}^{N}\Gamma(v_n(p_n))$.

(1.8) $\Gamma(v(p))>0$ *for any neighbourhood* $v(p)$.

(1.9) *For any subset A of R,* $\Gamma(A)=\inf_{\{v(p)\}}\sum_{v(p)\in\{v(p)\}}\Gamma(v(p))$, *where* $\{v(p)\}$ *is a family of neighbourhoods*[8] *whose union contains A.*

Then every open set is Γ*-measurable and we have* $\Gamma(A)>0$ *for every non-empty open set A.*

As the preparation for the proof of the theorem we give some lemmas.

Lemma 1.1. If an outer measure Γ satisfies (1.7) then we have (1.10) $\Gamma(\bigcup_{n=1}^{\infty} v_n(p_n))=\sum_{n=1}^{\infty}\Gamma(v_n(p_n))$ for every disjoint countable sequence of neighbourhoods.

Lemma 1.2. If an outer measure Γ satisfies (1.6)–(1.8) then, for every non-empty open set G, we have

$$\Gamma(G)=\sup_{\{v_n(p_n)\}}\sum_{n}\Gamma(v_n(p_n)), \tag{1.11}$$

where $\{v_n(p_n)\}$ is a finite disjoint family of neighbourhoods whose union is contained in G.

Lemma 1.3. Under the conditions (1.8) and (1.9), for every subset A of R, we have $\Gamma(A)=\inf_{G}\Gamma(G)$, where G is an open set which contains A.

Proof of Theorem 1. In virtue of Lemma 1.3, it is sufficient to prove that

$$\Gamma(U)\geq\Gamma(U\frown A)+\Gamma(U\frown(R-A)) \tag{1.12}$$

for arbitrary two open sets U and A. Let $\{v_n(p_n);\ 1\leq n\leq N\}$ be a family of disjoint finite neighbourhoods whose union is contained in $U\frown A$. We shall prove the inequality

$$\Gamma(U)\geq\sum_{n=1}^{N}\Gamma(v_n(p_n))+\Gamma(U\frown(R-\overline{\bigcup_{n=1}^{N}v_n(p_n)})). \tag{1.13}$$

If $U\frown(R-\overline{\bigcup_{n=1}^{N}v_n(p_n)})=0$, the proof is obvious. If $U\frown(R-\overline{\bigcup_{n=1}^{N}v_n(p_n)})\neq 0$, take any finite disjoint family $\{u_m(q_m);\ 1\leq m\leq M\}$ of neighbourhoods in it. Then, by (1.3) and (1.7), $\Gamma(U)\geq\sum_{n=1}^{N}\Gamma(v_n(p_n))+\sum_{m=1}^{M}\Gamma(u_m(q_m))$. So, by Lemma 1.2, we have (1.13).

While, from the relation $U\frown(R-A)\subseteq U\frown(R-\bigcup_{n=1}^{N}v_n(p_n))\subseteq\{U\frown(R-\overline{\bigcup_{n=1}^{N}v_n(p_n)})\}\smile\{U\frown(\overline{\bigcup_{n=1}^{N}v_n(p_n)}-\bigcup_{n=1}^{N}v_n(p_n))\}$, we have

$$\Gamma(U\frown(R-A))\leq\Gamma(U\frown(R-\overline{\bigcup_{n=1}^{N}v_n(p_n)})) \tag{1.14}$$

by (1.6).

Substituting (1.14) into (1.13) we obtain

$$\Gamma(U)\geq\sum_{n=1}^{N}\Gamma(v_n(p_n))+\Gamma(U\frown(R-A)).$$

Using Lemma 1.2, again we get (1.12). q.e.d.

2. Let R be an ω_0-ranked space.[9] For any subset A of R and

8) The power of $\{v(p)\}$ is not necessarily finite or countable.

9) A ranked space is called ω_0-ranked space if $\omega_\nu=\omega_0$. Cf. [K. Kunugi, I], Op. cit. We do not require, without contrary indication, F Hausdorff's axiom (C). Cf. H. Okano: Some operations on the ranked spaces. I, Proc. Japan Acad., **33**, 172–176 (1957).

a non-negative integer n, (n, A) denotes the upper limit of non-negative integers m such that there is a disjoint family of m neighbourhoods of rank n whose union is contained in A. If $(n, v(p))=(n, u(q))$ whenever $v(p), u(q)\in\mathfrak{V}_m$, we set $(n, m)=(n, v(p))$, where $v(p)\in\mathfrak{V}_m$. Then obviously we have the inequality

(2.1) $$(n, m)(m, l)\leq(n, l)$$

if they are determined.

In the sequel we assume that R is an ω_0-ranked space which satisfies the following conditions (2.2)–(2.4):

(2.2)[10] For every neighbourhood $v(p)$ of a point p there exists a positive integer n such that, for any integer $m, n\leq m$, there exists a neighbourhood $u(p)$ of rank m included in $v(p)$.

(2.3) For arbitrary two non-negative integers m and n, (m, n) is definitive.

(2.4) There exists a non-negative integer n_0 such that $1\leq(n, n_0)<+\infty$ if $n\geq n_0$.

From (2.1) and (2.4) we have $0\leq\dfrac{(n, m)}{(n, n_0)}\leq\dfrac{1}{(m, n_0)}\leq1$ if $n, m\geq n_0$. Therefore there exists an increasing sequence of integers $n_0<n_1<\cdots<n_k<\cdots$ such that, for every integer m $(m\geq n_0)$, the sequence $\left\{\dfrac{(n_k, m)}{(n_k, n_0)};\ k=0,1,\cdots\right\}$ is convergent. We set $\lambda(m)=\lim\limits_{k\to\infty}\dfrac{(n_k, m)}{(n_k, n_0)}$ and write $\lambda(v(p))=\lambda(m)$ for every neighbourhood $v(p)$ of rank m.

Now we define an outer measure in R as follows: We shall set $\mu^*(0)=0$ and, if $A\neq0$, $\mu^*(A)=\inf\limits_{\{v(p)\}}\sum\limits_{v(p)\in\{v(p)\}}\lambda(v(p))$, where $\{v(p)\}$ is a family of neighbourhoods[11] whose union covers A. Then the set function μ^* is an outer measure: μ^* satisfies the conditions (1.1)–(1.4).

Remark 1. If R is a left (right) ranked group[12] then μ^* is left (right) invariant, i.e. for every subset A of R and every point p of R, $\mu^*(pA)=\mu^*(A)$ $(\mu^*(Ap)=\mu^*(A))$.

Henceforce we shall study some properties of the outer measure μ^*. First we give the following lemma.

Lemma 2.1. If $\{v_m(p_m);\ m=1,2,\cdots\}$ is a finite or countable disjoint family of neighbourhoods whose union is contained in a neighbourhood $v(p)$, then we have $\sum\limits_m\lambda(v_m(p_m))\leq\lambda(v(p))$.

Now we shall investigate a property of the regularly complete ranked spaces.[13]

Definition 3.[13] An ω_0-ranked space R is called regularly complete

10) Cf. [H. Okano, Section 2 (a*)], Op. cit.

11) See 8). In the sequel we use the terminology "*neighbourhood*" only when it has a rank.

12) Cf. [H. Okano, Definition 4], Op. cit.

13) [K. Kunugi, I, Définition 4], Op. cit.

if, for every sequence of fundamental sequences $\{v_n^m(p_n^m)\}$ in R which satisfies the condition that $p_n^m=p_n^0$ and $v_n^m(p_n^m)=v_n^0(p_n^0)$ if $m\geq n$, there exist a point q in $\bigcap_n v_n^0(p_n^0)$ and a point q_m in $\bigcap_n v_n^m(p_n^m)$ such that $\lim_{m\to\infty} q_m=q$.

Lemma 2.2. Let R be an ω_0-ranked space which satisfies F. Hausdorff's axiom (C) and the following separation axiom (D$'$):

(D$'$) If $p\neq q$ there exist a neighbourhood $v(p)$ of p and a neighbourhood $u(q)$ of q such that $\overline{v(p)}\frown\overline{u(q)}=0$.

Then R is regularly complete if and only if, for every fundamental sequence $\{v_n(p_n)\}$, $\bigcap_n v_n(p_n)$ consists of one and only one element, say p, and, for any neighbourhood $v(p)$ of p, there exists a positive integer N such that $v_N(p_N)\subseteq v(p)$.

Definition 4. Let ε be a positive number and let $\{v_n(p_n);\ 1\leq n\leq N\}$ and $\{u_m(q_m);\ 1\leq m\leq M\}$ be two families of disjoint finite neighbourhoods. If $\sum_{n=1}^N \lambda(v_n(p_n))>\sum_{m=1}^M (u_m(q_m))-\varepsilon$ and, for each n, there exists a neighbourhood $v_n'(p_n)$ of p_n such that the rank of $v_n(p_n)>$ the rank of $v_n'(p_n)>\operatorname{Max}_m$ the rank of $u_m(q_m)$ and $v_n(p_n)\subseteq v_n'(p_n)\subseteq u_m(q_m)$ for some m, then $\{v_n(p_n)\}$ is called an ε-approximation of $\{u_m(q_m)\}$.

Theorem 2. *Let R be a regularly complete ω_0-ranked space with the conditions* (C) *and* (D$'$). *If μ^* satisfies* (1.6) *and the following condition* (2.5) *is satisfied:*

(2.5) *for any positive number ε and any neighbourhood $v(p)$ there exists an ε-approximation of $v(p)$,*

then every open set is μ^*-measurable and we have $\mu^*(v(p))=\lambda(v(p))$ for every neighbourhood $v(p)$.

Proof. First we remark that under the assumption of (2.5), for any ε and every finite disjoint family $\{v_n(p_n)\}$ of neighbourhoods, there exists an ε-approximation of $\{v_n(p_n)\}$.

If there is a positive integer m_0 such that $\lambda(m_0)=0$ then μ^* is identically zero. Therefore we assume that $\lambda(m)>0$ for every m. Now we shall show that, if $\{v_n(p_n);\ 1\leq n\leq N\}$ is a disjoint finite family of neighbourhoods and $\{u(q)\}$ is a family of neighbourhoods such that $\bigcup_{u(q)\in\{u(q)\}} u(q)\supseteq\bigcup_{n=1}^N v_n(p_n)$, then we have the relation

$$\sum_{n=1}^N \lambda(v_n(p_n))\leq\sum_{u(q)\in\{u(q)\}}\lambda(u(q)). \tag{2.6}$$

Let ε be any positive number. Set $\mathfrak{A}_0=\{v_n(p_n)\}$ and $\mathfrak{A}_0^*=$ the subfamily of $\mathfrak{A}_0$ which consists of neighbourhoods $v_n(p_n)$ such that $v_n(p_n)\subseteq u(q)$ for some $u(q)$ of $\{u(q)\}$. By induction we obtain two sequences $\mathfrak{A}_n$ and $\mathfrak{A}_n^*$, $n=0,1,\cdots$, such that $\mathfrak{A}_n$ is an $\varepsilon/2^n$-approximation of $\mathfrak{A}_{n-1}-\mathfrak{A}_{n-1}^*$ and $\mathfrak{A}_n^*$ is the subfamily of $\mathfrak{A}_n$ which consists of neighbourhoods $v(p)$ such that $v(p)\subseteq u(q)$ for some $u(q)$ of $\{u(q)\}$. Then from Lemma 2.1, we have $\sum_{v(p)\in\bigcup_n \mathfrak{A}_n^*} \lambda(v(p)) \le \sum_{u(q)\in\{u(q)\}} \lambda(u(q))$.

So we shall prove the relation

(2.7) $$\textstyle\sum_{n=1}^{N} \lambda(v_n(p_n))-\varepsilon < \sum_{v(p)\in\bigcup_n \mathfrak{A}_n^*} \lambda(v(p)).$$

Let $\mathfrak{J}$ be the totality of intervals I in the real line such that $I=[a,b)$,[14] $a<b$. And let φ be a correspondence of $\bigcup_{n=0}^{\infty}\mathfrak{A}_n$ into $\mathfrak{J}$ such that

(2.8) $\varphi(v(p))\supseteq\varphi(w(r))$ if and only if $v(p)\supseteq w(r)$,

(2.9) $\varphi(v(p))\cap\varphi(w(r))=0$ if and only if $v(p)\frown w(r)=0$

and

(2.10) $\nu[\varphi(v(p))]=\lambda(v(p))$, where ν is Lebesgue measure in the real line.[15]

We set $J_n=\bigcup_{v(p)\in\mathfrak{A}_{n-1}-\mathfrak{A}_{n-1}^*}\varphi(v(p))-\bigcup_{v(p)\in\mathfrak{A}_n}\varphi(v(p))$. Since $\nu(J_n)<\varepsilon/2^n$ and $\bigcup_{n=1}^{\infty}J_n$ is a sum of disjoint countable intervals, then $\nu(\bigcup_{n=1}^{\infty}J_n)<\varepsilon$. We set $Q=\bigcup_n\varphi(v_n(p_n))-\bigcup_{v(p)\in\bigcup_{n=0}^{\infty}\mathfrak{A}_n^*}v(p)-\bigcup_{n=1}^{\infty}J_n$. If $\sum_n\lambda(v_n(p_n))-\varepsilon \ge \sum_{v(p)\in\bigcup_{n=0}^{\infty}\mathfrak{A}_n^*}\lambda(v(p))$, then we have $\nu(Q)>0$ and therefore there exists a countable sequence of intervals $\varphi(v^0(p^0))\supseteq\varphi(v^1(p^1))\supseteq\cdots\supseteq\varphi(v^n(p^n))\supseteq\cdots$ such that

(2.11) $$v^n(p^n)\in\mathfrak{A}_n-\mathfrak{A}_n^*.$$

From the property of approximations there is a fundamental sequences $\{w_n(r_n)\}$ such that $r_{2n-1}=p^n$ and $w_{2n-1}(r_{2n-1})=v^n(p^n)$. Since R is regularly complete, there exists a point r in $\bigcap_n v^n(p^n)$. While r is contained in some $u(q)$ and therefore there exists a neighbourhood $w(r)$ of r which is included in $u(q)$. From Lemma 2.2, we get $v^n(p^n)\subseteq w(r)$ for some $v^n(p^n)$ and it contradicts (2.11). Hence we have (2.7) and therefore (2.6).

From (2.6), for every disjoint finite family $\{v_n(p_n)\}$ of neighbourhoods, we have $\mu^*(\bigcup_n v_n(p_n))=\sum_n\lambda(v_n(p_n))$ and especially $\mu^*(v(p))=\lambda(v(p))$. The properties (1.7)-(1.9) of Theorem 1 are immediate results of this fact. Hence, by Theorem 1, the proof is concluded.

3. Examples

Example 1. Let R be n-dimensional Euclidian space. Since $\omega(R^n)=\omega_0$, we set $\mathfrak{B}_m=$the totality of $v_m(p=(p_i))=\left\{x=(x_i);\ \operatorname{Max}_i|x_i-p_i|<\frac{1}{m+2}\right\}$. And put $n_0=0$ in (2.4) then μ^* coincides with Lebesgue measure.

Example 2. Let R be the ranked space N_+ of [K. Kunugi, I,

14) $[a,b)$ denotes the set of all x such that $a\le x<b$.

15) There exists always at least one correspondence like that.

Exemple 2].[16] Set $n_0=1$ in (2.4) and we get 1-dimensional Lebesgue measure as μ^*.

Example 3. Let R be the ranked space PU of [K. Kunugi, II, Exemple 3].[16] μ^* coincides essentially with 2-dimensional Lebesgue measure.

Example 4. If R is a discrete ranked space, then $\mu^*(A)=$the number of elements of A if A is finite and $\mu^*(A)=+\infty$ if A is infinite.

Example 5. Let $R=P_{i\in I}(R_i=R^1)$ be an infinite Cartesian product of real lines. For every positive integer n and every subset $\{i_1,\cdots,i_n\}$ of n elements of I, $V^n_{\{i_1,\dots,i_n\}}$ denotes the set of all $p=(p_i)$ such that $\frac{1}{n}>p_{i_k}\geq 0$ for every $k(1\leq k\leq n)$ and $1>p_i\geq 0$ for every i. The system of neighbourhoods of the origin is the totality of such $V^n_{\{i_1,\dots,i_n\}}$ and the neighbourhoods of another point are given by the translation. Then we have $\omega(R)=\omega_0$ and we set $\mathfrak{B}_n=$the family if all neighbourhoods $V^n_{\{i_1,\dots,i_n\}}$ of all points. Then R is an ω_0-ranked space which satisfies the conditions (2.2)-(2.4) and therefore we can construct μ^*. R does not satisfy F. Hausdorff's axiom (C).

Remark 2. The spaces of Examples 2, 3 and 5 are ranked groups but not topological groups.

16) See 1).