

28. Note on Idempotent Semigroups. IV. Identities of Three Variables¹⁾

By Naoki KIMURA

Tokyo Institute of Technology and Tulane University

(Comm. by K. SHODA, M.J.A., March 12, 1958)

§ 1. *Introduction.* In this short note we shall present the classification of all identities of three variables on idempotent semigroups.

The motivation of taking three for the number of variables has come from the fact that many important identities on idempotent semigroups are written by three or fewer independent variables.

Here only the main two theorems and necessary definitions are given, and the proofs are all omitted. We will study them in detail elsewhere.²⁾

§ 2. *The classification theorem.* Let $X = \{x, y, z\}$. Let F be the free semigroup generated by X . Then an identity is a pair of element of F , say (P, Q) , for which we use the notation $P=Q$.

We shall say that an identity $P=Q$ is equivalent to an identity $P'=Q'$, if the following conditions are satisfied:

(1) If S is an idempotent semigroup such that for every homomorphism $h: F \rightarrow S$ always $h(P)=h(Q)$, then also $h(P')=h(Q')$.

(2) If S is an idempotent semigroup such that for every homomorphism $h: F \rightarrow S$ always $h(P')=h(Q')$, then also $h(P)=h(Q)$.

Theorem 1. *Any identity of three variables on idempotent semigroups is equivalent to one of the following 18 distinct properties or identities.*

- | | |
|-----------------------------------|---------------|
| (1) <i>triviality,</i> | $x=y;$ |
| (2) <i>left singularity,</i> | $xy=x;$ |
| (3) <i>right singularity,</i> | $xy=y;$ |
| (4) <i>rectangularity,</i> | $xyx=x;$ |
| (5) <i>commutativity,</i> | $xy=yx;$ |
| (6) <i>left regularity,</i> | $xyx=xy;$ |
| (7) <i>right regularity,</i> | $xyx=yx;$ |
| (8) <i>universality,</i> | $x=x;$ |
| (9) <i>left normality,</i> | $xyz=xzy;$ |
| (10) <i>right normality,</i> | $xyz=yxz;$ |
| (11) <i>normality,</i> | $xyzx=xzyx;$ |
| (12) <i>regularity,</i> | $xyzx=xyxzx;$ |
| (13) <i>left quasi-normality,</i> | $xyz=xzyz;$ |

1) This work was partially supported by the National Science Foundation, U. S. A.

2) This is an abstract of the paper which will appear elsewhere.

- (14) *right quasi-normality*, $xyz = xyxz$;
 (15) *left semi-normality*, $xyzx = xzyzx$;
 (16) *right semi-normality*, $xyzx = xyzyx$;
 (17) *left semi-regularity*, $xyzx = xyxzyzx$;
 (18) *right semi-regularity*, $xyzx = xyzxyzx$.

For the structures of idempotent semigroups satisfying identities above, see [1] for (6), (7) and (12), [2] for (9), (10) and (11), and [3] for (13)–(18).

§ 3. *The criterion theorem.* Let $P \equiv x_1 x_2 \cdots x_n$, where $x_i \in X$. Then $x_1(x_n)$ is called the *head (tail)* of P . The initial part of P is an ordered subset of X which is obtained by deleting all variables of P except the first occurrence reading from the left. The main left factor of P is the shortest cut of P containing all variables in it. Thus if $P \equiv xxyxyzxy$, then its initial part is xyz and its main left factor is $xxyxyz$. Then final part and the main right factor are dually defined. If P and Q have the same initial (final) part, then we call them *coinitial (cofinal)*.

Let $P=Q$ be an identity. If the main left (right) factors of P and Q are sent to the same element by any homomorphism from F to any idempotent semigroup, then we call them *left (right) similar*.

Let $P=Q$ be an identity. It is called *homotypical* if P and Q have the same set of variables. Otherwise, it is called *heterotypical*. If P and Q contain all three variables, then the identity is called *general*. It is clear that a general identity is necessarily homotypical.

Theorem 2. *An identity $P=Q$ of three variables is equivalent to*

- (1) *triviality if and only if it is heterotypical and P and Q have different heads and different tails;*
- (2) *left singularity if and only if it is heterotypical and P and Q have the same head but different tails;*
- (3) *right singularity if and only if it is heterotypical and P and Q have the same tail but different heads;*
- (4) *rectangularity if and only if it is heterotypical and P and Q have the same head and the same tail;*
- (5) *commutativity if and only if it is homotypical and P and Q have different heads and different tails;*
- (6) *left regularity if and only if (it is homotypical and) P and Q are coinitial and have different tails;*
- (7) *right regularity if and only if (it is homotypical and) P and Q are cofinal and have different heads;*
- (8) *universality if and only if (it is homotypical and) P and Q are both left and right similar;*
- (9) *left normality if and only if it is general and P and Q have the same head and different tails but are not coinitial;*

- (10) *right normality if and only if it is general and P and Q have the same tail and different heads but are not cofinal;*
- (11) *normality if and only if it is general and P and Q have the same head and the same tail but are neither coinitial nor cofinal;*
- (12) *regularity if and only if it is general and P and Q are both coinitial and cofinal but are neither left nor right similar;*
- (13) *left quasi-normality if and only if it is general and P and Q have the same head and are cofinal but are neither coinitial nor right similar;*
- (14) *right quasi-normality if and only if it is general and P and Q have the same tail and are coinitial but are neither cofinal nor left similar;*
- (15) *left semi-normality if and only if it is general and P and Q have the same head and are right similar but not coinitial;*
- (16) *right semi-normality if and only if it is general and P and Q have the same tail and are left similar but not cofinal;*
- (17) *left semi-regularity if and only if it is general and P and Q are right similar and coinitial but not left similar;*
- (18) *right semi-regularity if and only if it is general and P and Q are left similar and cofinal but not right similar.*

References

- [1] Naoki Kimura: Note on idempotent semigroups. I, Proc. Japan Acad., **33**, 642 (1957).
- [2] Miyuki Yamada and Naoki Kimura: Ditto. II, Proc. Japan Acad., **34**, 110 (1958).
- [3] Naoki Kimura: Ditto. III, Proc. Japan Acad., **34**, 113 (1958).