# 48. *Measures in the Ranked Spaces. II*

By Hatsuo OKANO
Osaka University

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In the preceding paper[1] we showed a method to construct outer measures in ranked spaces. But, in general, every open set is not always measurable. So in this note assuming some conditions we give a method to construct Borel measures in ranked spaces: By a Borel measure in an $\omega_0$-ranked space $R$ which satisfies F. Hausdorff's axiom (C) we mean a finite or infinite real valued, non-negative, and countably additive set function, defined on the countably additive class of sets, denoted by $\mathfrak{B}$, generated by the class of all open sets.

1. Definition 1. For two neighbourhoods $v(p)$ and $u(q)$ in an $\omega_0$-ranked space we call that $v(p)$ is *strongly* contained in $u(q)$, denoted by $v(p)\Subset u(q)$, if there exists a neighbourhood $v'(p)$ of $p$ such that $v(p)\subseteq v'(p)\subseteq u(q)$ and the rank of $v(p)>$the rank of $v'(p)>$the rank of $u(q)$. A disjoint finite family of neighbourhoods $\{v_n(p)\}$ is called a *packing* of a neighbourhood $u(q)$ if $v_n(p_n)\Subset u(q)$ for each $n$. And let $\{v_n(p_n)\}$ and $\{u_m(q_m)\}$ be two packings of $v(p)$ and $u(q)$ respectively. We call that the packings have the *same type* if, for each rank $n$, the number of neighbourhoods of rank $n$ of $\{v_n(p_n)\}$ coincides with that of $\{u_m(q_m)\}$.

Let $R$ be an $\omega_0$-ranked space which satisfies the following conditions (1.1) - (1.4):[2]

(1.1) For every neighbourhood $v(p)$ of a point $p$ there exists a rank $n$ such that, for any rank $m$, $m\geq n$, there exists a neighbourhood $u(p)$ of rank $m$ included in $v(p)$.[3]

(1.2) There is a rank $n_0$ such that, for any rank $n$, the upper limit of numbers of disjoint neighbourhoods of rank $n$ contained strongly in a neighbourhood of rank $n_0$ is finite.[4]

(1.3) For two neighbourhoods $v(p)$ and $u(q)$ of the same rank and a packing of $v(p)$, $u(q)$ has a packing of the same type.[5]

(1.4) For any fundamental sequence $\{v_n(p_n)\}$ there exists a point $p$ in $\bigcap_n v_n(p_n)$ such that, for any neighbourhood $v(p)$ of $p$, there exists

1) H. Okano: Measures in the ranked spaces, Proc. Japan Acad., **34**, 136-141 (1958), cited by [I] in this paper.

2) In the sequel we use the terminology *neighbourhood* only when it has a rank.

3) Cf. [I, (2.2)].

4) Cf. [I, (2.4)].

5) Cf. L. H. Loomis: Haar measure in uniform structure, Duke Math. J., **16**, 193-208 (1949).

an $N$ such that $v_N(p_N) \subseteq v(p)$.[6)]

From the condition (1.3), for any pair of ranks $n$ and $m$, we denote the upper limit of numbers of disjoint neighbourhoods of rank $n$ contained strongly in a neighbourhood of rank $m$ by $[n, m]$. Then we have

(1.5) $$0 \leq [n, m] \leq +\infty,$$

(1.6) $$[n, m][m, l] \leq [n, l]$$

and

(1.7) there exists a rank $m_0$ such that $1 \leq [n, n_0] < +\infty$ if $n \geq m_0$.

Therefore we get $0 \leq \frac{[n, m]}{[n, n_0]} \leq \frac{1}{[m, n_0]} \leq 1$ if $n, m \geq m_0$ and, hence, there exists an increasing sequence of integers $m_0 < n_1 < \cdots < n_k < \cdots$ such that, for every rank $m$ $(m \geq m_0)$, the sequence $\left\{\frac{[n_k, m]}{[n_k, n_0]};\ k=1, 2, \cdots\right\}$ is convergent. We set $\lambda(m) = \lim_{k\to\infty} \frac{[n_k, m]}{[n_k, n_0]}$ and put $\lambda(v(p)) = \lambda(m)$ for every neighbourhood $v(p)$ of rank $m$ $(m \geq m_0)$.[7)] Then $\lambda$ is a set function, defined on the class of neighbourhoods $\mathfrak{V} = \bigcup_{m=m_0}^{\infty} \mathfrak{V}_m$, such that

(1.8) $$0 \leq \lambda(v(p)) \leq 1,$$

(1.9) for arbitrary two neighbourhoods $v(p)$ and $u(q)$ of the same rank, $\lambda(v(p)) = \lambda(u(q))$

and

(1.10) if $\{v_n(p_n)\}$ is a packing of $u(q)$ then we have $\lambda(u(q)) \geq \sum_n \lambda(v_n(p_n))$.

Now we put $\lambda_0 = \lambda$. And suppose that we have already defined the functions $\lambda_\beta$ on $\mathfrak{V}$ for all $\beta$ such that $0 \leq \beta < \alpha$ where $\alpha < \Omega$[8)] and they satisfy the following conditions (1.11) - (1.15):

(1.11) $$1 \geq \lambda_0(v(p)) \geq \lambda_1(v(p)) \geq \cdots \geq \lambda_\beta(v(p)) \geq \cdots \geq 0.$$

(1.12) $\lambda_\beta(v(p)) = \lambda_\beta(u(q))$ for arbitrary two neighbourhoods $v(p)$ and $u(q)$ of the same rank.

(1.13) $\lambda_\beta(v(p)) \geq \sum_n \lambda_\beta(v_n(p_n))$ for every packing $\{v_n(p_n)\}$ of $v(p)$.

(1.14) If $\beta$ is an isolated ordinal number then $\lambda_\beta(v(p)) = \sup_{\{v_n(p_n)\}} \sum_n \lambda_{\beta-1}(v_n(p_n))$, where $\{v_n(p_n)\}$ is a packing of $v(p)$.

(1.15) If $\beta$ is a limiting ordinal number then $\lambda_\beta(v(p)) = \lim_{k\to\infty} \lambda_{\gamma_k}(v(p))$, where $\{\gamma_k\}$ is an increasing countable sequence of ordinal numbers such that $\lim_{k\to\infty} \gamma_k = \beta$.

Then we define $\lambda_\alpha$ by (1.14) or (1.15) if $\alpha$ is isolated or limiting respectively. Thus we obtain an $\Omega$-sequence $\{\lambda_\alpha; 0 \leq \alpha < \Omega\}$ of functions on $\mathfrak{V}$ satisfying (1.11) - (1.15).

6) This condition, in general, is more restrictive than regular completeness but, if $R$ satisfies the axioms (C) and (D′), then the both conditions coincide. Cf. [I, Lemma 2.2].

7) This function $\lambda$ differs, in general, with that defined in [I].

8) $\Omega$ denotes the first uncountable ordinal number.

By (1.12) we set $\lambda_\alpha(m)=\lambda_\alpha(v(p))$, where $v(p)\in\mathfrak{B}_m$. From (1.11), for every rank $m$, there exists $\alpha(m)<\Omega$ such that $\lambda_{\alpha(m)}(m)=\lambda_{\alpha(m)+1}(m)=\cdots=\lambda_\alpha(m)=\cdots$ for every $\alpha$ such that $\alpha(m)\leq\alpha<\Omega$. Since $\Omega$ is an inaccessible ordinal number, then $\sup_m \alpha(m)<\Omega$. Put $\bar{\alpha}=\sup_m \alpha(m)$ and then we get $\lambda_{\bar{\alpha}}(v(p))=\lambda_{\bar{\alpha}+1}(v(p))=\cdots$ for every $v(p)$. We denote the constant by $\lambda_\Omega(v(p))$. Then we have the following

Theorem 1. *$\lambda_\Omega(v(p))$ is a finite real valued set function, defined on $\mathfrak{B}$, such that*

(1.16) $0<\lambda_\Omega(v(p))\leq 1$ *except the case that it is identically zero,*

(1.17) $\lambda_\Omega(v(p))=\lambda_\Omega(u(q))$ *for any pair of neighbourhoods $v(p)$ and $u(q)$ of the same rank*

*and*

(1.18) $\lambda_\Omega(v(p))=\sup_{\{v_n(p_n)\}}\sum_n \lambda_\Omega(v_n(p_n))$, *where $\{v_n(p_n)\}$ is a packing of $v(p)$.*

2. Let $\mathfrak{O}$ be the class of all open sets of $R$. We set $\lambda_*(0)=0$ for the empty set 0 and, for every non-empty open set $G$, $\lambda_*(G)=\sup_{\{v_n(p_n)\}}\sum_n \lambda_\Omega(v_n(p_n))$, where $\{v_n(p_n)\}$ is a disjoint finite family of neighbourhoods each of which is contained in $G$.

Theorem 2. *$\lambda_*$ is a finite or infinite real valued, non-negative, monotone, countably subadditive and countably additive set function, defined on $\mathfrak{O}$ such that $\lambda_*(0)=0$ and, for every non-empty open set $G$, $\lambda_*(G)>0$ except the case that $\lambda_\Omega$ is identically zero.*

The subadditivity of the set function $\lambda_*$ is proved by an analogous way with the proof of Theorem 2 of the preceding paper from the conditions (1.4) and (1.18).

3. For every subset $A$ of $R$ we set $\nu^*(A)=\inf_G \lambda_*(G)$, where $G$ is an open set which contains $A$. Then $\nu^*$ is an outer measure in $R$, i.e. a finite or infinite real valued, non-negative, monotone and countably subadditive set function defined on the class of all subsets of $R$ such that $\nu^*(0)=0$, and satisfies the conditions

(3.1) $\nu^*(G)=\lambda_*(G)$ for any open set $G$

and

(3.2) $\nu^*(A\smile B)=\nu^*(A)+\nu^*(B)$ if there exist two open sets $G$ and $H$ such that $A\subseteq G$, $B\subseteq H$ and $G\frown H=0$.

Let $\mathfrak{L}^*$ denote the class of all $\nu^*$-measurable sets.

Lemma. If, for every neighbourhood $v(p)$, there exists a closed set $F$ such that $F\supseteq v(p)$ and $\nu^*(F-v(p))=0$ then every open set is $\nu^*$-measurable and therefore we have $\mathfrak{L}^*\supseteq\mathfrak{B}$.

From this lemma we obtain the following

Theorem 3. *We set $\nu(A)=\nu^*(A)$ for every subset $A$ belonging to $\mathfrak{L}^*$. Then, if the hypothesis of the above lemma is satisfied, $\nu$ is a Borel measure in $R$.*[9] *And we have $\nu(G)>0$ for every non-empty open set $G$ except the case that $\lambda_\Omega$ is identically zero.*

9) We do not assume the axiom (C).