45. On Quasi-continuous Mappings Defined on a Product Space

By Yoshimichi MIBU

(Comm. by Z. SUETUNA, M.J.A., April 12, 1958)

Let $f(x, y)$ be a function of two real variables. H. Hahn proved that if 1) for any fixed x, the function $f(x, y)$ is a continuous function of one variable y, and 2) for any fixed y, the function $f(x, y)$ is also a continuous function of x, then the set of continuity points of the function $f(x, y)$ is dense in the plane. Our purpose is to extend Hahn's theorem.

Definition and notation. Let X be a topological space and M a metric space. Suppose that $f(x)$ is a mapping of X into M. If the set of discontinuity points of $f(x)$ is of the first category, then $f(x)$ is called a quasi-continuous mapping of X into M.

Let E be a subset of M. By $\delta(E)$ we shall denote the diameter of the set E. We set

$$\omega(f; x)=\inf_{U(x)} \delta(f(U(x))),$$

where $U(x)$'s are neighborhoods of x.

Remark 1. Let $f(x)$ be a mapping of X into M. In order that $f(x)$ be continuous at a point x_0 it is necessary and sufficient that $\omega(f; x_0)=0$ holds. Hence the set of the discontinuity points of $f(x)$ coincides with the set $\bigcup_{n=1}^{\infty} \{x; \omega(f; x)\geqq 1/n\}$. It is easily seen that each set $\{x; \omega(f; x)\geqq 1/n\}$ is a closed set of X for every n.

Remark 2. Let $f(x)$ be a quasi-continuous mapping of X into M. If every open set of X is of the second category, then clearly the set of the continuity points of $f(x)$ is dense in X.

Theorem 1. Let X and Y be two topological spaces and M a metric space. And let $f(x, y)$ be a mapping of the product space $X\times Y$ into M. We assume that the following conditions are satisfied:

1) For any fixed $x\in X$, the mapping $f(x, y)$ is a continuous mapping of Y into M.

2) There exists a set H which is dense in the space Y and $f(x, y)$ is a continuous mapping of the space X into M for any fixed $y\in H$.

3) Every open subset of the space X is of the second category.

4) The space Y satisfies the first axiom of countability. Then $f(x, y)$ is a quasi-continuous mapping of the product space $X\times Y$ into M.

Proof. The mapping $f(x, y)$ can be regarded as a mapping $f(Z)$ of a single space $P=X\times Y$ into M. We set

(1) $$A_n=\{z;\ \omega(f;\ z)\geqq 1/n\}.$$

For the proof it is enough to show that the set A_n is a non-dense set for every n (see Remark 1). Assume that for an n_0 the set A_{n_0} is not non-dense. Since A_{n_0} is closed (see Remark 1), the set $A_{n_0}^i$ (interior of A_{n_0}) is not empty. Hence we set

(2) $$G=A_{n_0}^i \neq 0.$$

As G is an open set of the product space $P=X\times Y$, there exist open sets $U_0\subseteq X$ and $V_0\subseteq Y$ such that

(3) $$\{(x, y);\ x\in U_0,\ y\in V_0\}=U_0\times V_0\subseteq G.$$

We select a point $y_0\in V_0\frown H$ (see condition 2) of the theorem) and let $V_1, V_2,\cdots, V_n,\cdots$ be a complete system of neighborhoods of the point y_0. Without losing the generality we may assume that each V_i is contained in V_0.

(4) $$V_1, V_2,\cdots, V_n,\cdots,\ y_0\in V_i\subseteq V_0,\ i=1, 2,\cdots\ .$$

For the sake of convenience the mapping $f(x, y)$ will be denoted by $f_x(y)$ if $f(x, y)$ is regarded as a mapping of the space Y into M for any fixed x. Let ε be a positive number such that

(5) $$7\varepsilon < 1/n_0.$$

We set

(6) $$B_n=\{x;\ \delta(f_x(V_n))<\varepsilon\}.$$

Since for every $x\in X$ the mapping $f_x(y)$ is continuous at the point y_0, it is easily seen that $\bigcup_{n=1}^{\infty} B_n=X$. Hence if we set

(7) $$D_n = B_n\frown U_0,$$

then we have clearly $\bigcup_{n=1}^{\infty} D_n=U_0$. From condition 3) of the theorem, the open set U_0 is of the second category. And so there exists a natural number N such that the set D_N is not non-dense. Hence the set D_N^{ai} (interior of the closure of D_N) is not empty. Setting

(8) $$U = D_N^{ai}\frown U_0,$$

it is easily seen that $U\neq 0$. Now for the sake of convenience the mapping $f(x, y)$ will be denoted by $f_y(x)$ if $f(x, y)$ is regarded as a mapping of the space X into M for any fixed y. Since $f_{y_0}(x)$ is a continuous mapping of the space X into M, there exists a neighborhood U_1 such that

(9) $$\delta(f_{y_0}(U_1))<\varepsilon,\quad U_1\subset U.$$

We set

(10) $$W=U_1\times V_N(\subseteq U\times V_0\subseteq U_0\times V_0\subseteq G=A_{n_0}^i).$$

For two arbitrary points $(x, y)\in W$ and $(x', y')\in W$, we shall estimate the distance of two points $f(x, y)$ and $f(x', y')$. Since $f_x(y)$ is a continuous mapping of the space Y into M, there exists a point y_1 such that

(11) $$\rho(f(x, y),\ f(x, y_1))<\varepsilon,\quad y_1\in H\frown V_N.$$

On the other hand the mapping $f_{y_1}(x)$ is a continuous mapping of the space X into M. Hence there exists a point x_1 such that

(12) $$\rho(f(x, y_1),\ f(x_1, y_1))<\varepsilon, \quad x_1 \in U_1 \frown D_N.$$

Since $x_1 \in D_N$ and $y_0, y_1 \in V_N$, we have

(13) $$\rho(f(x_1, y_1),\ f(x_1, y_0))<\varepsilon. \quad \text{(See (6) and (7).)}$$

Quite similarly we can see that there exist two points y_1' and x_1' such and

(14) $$\rho(f(x', y'),\ f(x', y_1'))<\varepsilon, \quad y_1' \in H \frown V_N,$$

(15) $$\rho(f(x', y_1'),\ f(x_1', y_1'))<\varepsilon, \quad x_1' \in U_1 \frown D_N,$$

(16) $$\rho(f(x_1', y_1'),\ f(x_1', y_0))<\varepsilon.$$

On the other hand $x_1, x_1' \in U_1$, hence from (9) we have

(17) $$\rho(f(x_1, y_0),\ f(x_1', y_0))<\varepsilon.$$

From these inequalities (11)-(17) we have at once

(18) $$\rho(f(x, y),\ f(x', y'))<7\varepsilon.$$

Thus we have $\omega(f; (x, y)) \leqq 7\varepsilon$ for any point $(x, y) \in W$.

On the other hand $W \subseteq A^i_{n_0}$ (see (10)), and so we have $\omega(f; (x, y)) \geqq 1/n_0$. But by (5) $1/n_0 > 7\varepsilon$, so that we have arrived at a contradiction.

Theorem 2. Let X, Y, M, and $f(x, y)$ be as in Theorem 1. Suppose that the following conditions are satisfied:

1) There exists a subset $L \subseteq X$ which is of the first category and $f(x, y)$ is a quasi-continuous mapping of Y into M for any fixed $x \in X-L$.

2) For any fixed $y \in Y$, $f(x, y)$ is a continuous mapping of the space X into M.

3) Every open subset of the spaces X and Y is of the second category.

4) The space Y satisfies the second axiom of countability. Then the mapping $f(x, y)$ is a quasi-continuous mapping of the product space $X \times Y$ into M.

Proof. In the proof of the above theorem, we selected a complete system of neighborhoods of the point y_0 (see (4)). But in this theorem we must select a countable basis of the relative subspace V_0. Then we shall find that the other arguments are quite similar to those of the preceding theorem. So we shall omit the detailed proof.

Corollary. Let $f(x_1, x_2, \cdots, x_n)$ be a complex (real) valued function of n variables $x_1, x_2, \cdots, x_n$, where x_i's are complex (real) variables. Suppose that the following conditions are satisfied:

1) For any fixed system $(x_2, x_3, \cdots, x_n)$, the function $f(x_1, x_2, \cdots, x_n)$ is a quasi-continuous function of one variable x_1.

2) For any i $(2 \leqq i \leqq n)$, the function $f(x_1, x_2, \cdots, x_n)$ is a continuous function of one variable x_i for any fixed system $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$.

Then the function $f(x_1, x_2, \cdots, x_n)$ is a quasi-continuous function of n variables.

Application 1. Let G be an abstract group. Further we assume that G is also a complete metric space and the following condition is satisfied:

1) $\lim_{n\to\infty} x_n = x$ implies $\lim_{n\to\infty} x_n y = xy$ and $\lim_{n\to\infty} y x_n = yx$.

Then we have the following:

2) $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ imply $\lim_{n\to\infty} x_n y_n = xy$.

Proof. To each point $(x, y) \in G \times G$ we correspond a point $f(x, y) = xy \in G$. Then it is easily seen that the conditions 1), 2), 3), and 4) of Theorem 1 are all satisfied. (In this case $X = Y = M = G$.) Hence by Theorem 1, there exists a continuity point (x_0, y_0) of the mapping $f(x, y) = xy$. (Notice that the product space $G \times G$ is of the second category.) Suppose that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. From condition 1) we have $\lim_{n\to\infty} x_0 x^{-1} \cdot x_n = x_0 x^{-1} \cdot x = x_0$ and $\lim_{n\to\infty} y_n \cdot y^{-1} y_0 = y \cdot y^{-1} y_0 = y_0$. Since the point (x_0, y_0) is a continuity point of the mapping $f(x, y)$, we have $\lim_{n\to\infty} x_0 x^{-1} x_n \cdot y_n y^{-1} y_0 = x_0 y_0$. From this and condition 1) we have

$$\lim_{n\to\infty} x_n y_n = \lim_{n\to\infty} x x_0^{-1} \cdot x_0 x^{-1} x_n y_n y^{-1} y_0 \cdot y_0^{-1} y = x x_0^{-1} \cdot x_0 y_0 \cdot y_0^{-1} y = xy.$$

Application 2. Let E be a space of type (F) (see S. Banach: Théorie des opérations linéaires, p. 35). If $\lim_{n\to\infty} \lambda_n = \lambda$ and $\lim_{n\to\infty} x_n = x$, then we have $\lim_{n\to\infty} \lambda_n x_n = \lambda x$.

Proof. Let R be the one-dimensional euclidean space and $R \times E$ the product sapce of R and E. To each point $(\lambda, x) \in R \times E$ we correspond a point $\lambda x \in E$. Then we have a mapping $f(\lambda, x) = \lambda x$ of the product space $R \times E$ into E. It is easily seen that the conditions 1)–4) of Theorem 1 are all satisfied. Hence by Theorem 1 our assertion is proved quite similarly as in the above Application 1.