62. *Motion of the Ninth Satellite of Jupiter*

By Gen-ichiro HORI

Department of Astronomy, University of Tokyo

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1. **Introduction.** In a preceding paper[1] we have calculated an intermediate orbit of the ninth satellite of Jupiter, J-IX. The present work is its continuation and contains the calculation of a general orbit of J-IX, together with the method and the results of comparison between the calculated orbit and the observations and, at the same time, the determination of the values of angular constants of integration. The calculation of the general orbit is generally based on the theory of E. W. Brown and D. Brouwer.[2] Some devices are attempted, however, in the details of computation.

The comparison with observations undertaken in the present work is rather of a preliminary character. The results, however, appear to indicate that the theory of Brown-Brouwer is probably available for explaining the motion of J-IX, one of the most complicated motions in the solar system, and that the numerical values of the mean elements of J-IX deduced from observations of S. B. Nicholson[3] are correct in the range of accuracy. In order to verify the availability of the theory and to determine the numerical values of the mean elements to a higher degree of accuracy, it is necessary to go into details and to make a comparison referred not to the perturbations of the elements as in the present work, but directly to the space coordinates of J-IX. Only a brief summary of this work is given here. A more detailed description will be found in our thesis to be published shortly.

2. **Equations of variations; definitions of perturbations.** The general orbit is obtained by adding to the intermediate orbit such additional parts as depend on the inclination (θ-terms), the first power of the ratio of the parallaxes (parallactic terms), and the first power of the eccentricity of Jupiter (e'-terms).

The calculation of the additional parts is carried out by the use of the equations of variations. Such equations are easily derived from the original equations of motion by neglecting the squares and the products of the variations of the variables:

$$D\delta Q=\delta_1 Q'+Q'_Q\delta Q+Q'_U\delta U+Q'_v\delta(v-v'),$$

$$[D^2+(1-\varpi_1^*)^2]\delta U=\delta_1 U'+U'_Q\delta Q+U''_Q D\delta Q+U'''_Q D^2\delta Q+U'_U\delta U+U'_v\delta(v-v'),$$

$$D\delta(v-v')=\delta_1 v'+\Gamma_0 V'_\Gamma\frac{\delta\Gamma}{\Gamma_0}+V'_U\delta U+V'_v\delta(v-v'),\quad \frac{\delta\Gamma}{\Gamma_0}=\frac{1}{2}\delta Q-Z,$$

$$n\delta t=-\frac{2}{U_0^3}\delta U,\quad \delta v=\delta(v-v')+n'\delta t,\quad D\delta\theta=\frac{1}{\Gamma_0}D\delta v-\left(\frac{\partial T}{\partial\Gamma}\right)_0\frac{\delta\Gamma}{\Gamma_0},$$

where $Q=ln\,q$, $U=uq^{-\frac{1}{4}}$, $T=\dfrac{qR}{u^2}$, $Z=D^{-1}\left(\Gamma_0^{-1}\dfrac{\partial T}{\partial\theta_0}\right)$, and δ denotes the increments (variations) due to an addition $\delta_1 R$ to R, or $\delta_1 T$ to T.

For the variables contained in $\delta_1 Q'$, Q'_Q, etc., we substitute their values obtained in the intermediate orbit. These values are denoted by the suffix zero. The coefficients of δQ, δU, etc. are functions of the intermediate orbit only and of the form $\sum A_{\sin}^{\cos}(2\xi j+il)$, while $\delta_1 Q'$, $\delta_1 U'$, etc., containing $\delta_1 T$, are of different forms according as the form of $\delta_1 T$ consisting of the θ-terms, or parallactic terms, or e'-terms. δQ, δU, $\delta\Gamma/\Gamma_0$, and $\delta(v-v')$ are obtained from the first four of the preceding equations by successive approximations, and then δv, $\delta v'$ are separated by the fifth and the sixth equations. Finally $\delta\theta$ is obtained from the last equation. In the case of the θ-terms, however, this equation for $\delta\theta$ is incomplete in accuracy. This case will be treated in Sec. 4. A larger part of the computation can be decomposed to such operations as $Q'_Q\times\delta Q$, that is, to the products of two Fourier series.

3. **The disturbing function.** With our definition of T and U, the disturbing function can be written, by omitting the term $\dfrac{m'}{r'}$,

$$T=\frac{m'}{\mu}U^{-4}\left\{\frac{a^3}{r'^3}\left(\frac{3}{2}S^2-\frac{1}{2}\right)+q^{-\frac{1}{4}}U^{-1}\frac{a^4}{r'^4}\left(\frac{5}{2}S^3-\frac{3}{2}S\right)\right\},$$

with
$$S=\left(1-\frac{1}{2}\Gamma\right)\cos(v-v')+\frac{1}{2}\Gamma\cos(v+v'-2\theta).$$

By developing S^2 and S^3, we obtain

$$T=T_0+T_\theta+T_\alpha=T_{00}+T_\theta+T_\alpha+T_\eta,$$

where
$$T_0\div\frac{m'}{\mu}U^{-4}=\frac{3}{4}\frac{a^3}{r'^3}\left\{\left(\frac{1}{3}-\Gamma+\frac{1}{2}\Gamma^2\right)+\left(1-\frac{1}{2}\Gamma\right)^2\cos(2v-2v')\right\},$$

$$T_\theta\div\frac{m'}{\mu}U^{-4}=\frac{3}{4}\frac{a^3}{r'^3}\left[\left(1-\frac{1}{2}\Gamma\right)\Gamma\{\cos(2v-2\theta)+\cos(2v'-2\theta)\}\right.$$
$$\left.+\frac{1}{4}\Gamma^2\cos(2v+2v'-4\theta)\right],$$

$$T_\alpha\div\frac{m'}{\mu}U^{-4}=q^{-\frac{1}{4}}U^{-1}\frac{a^4}{r'^4}\left[\frac{5}{8}\left(1-\frac{1}{2}\Gamma\right)^3\cos(3v-3v')\right.$$
$$\left.+\left(\frac{3}{8}-\frac{33}{16}\Gamma+\frac{75}{32}\Gamma^2-\frac{45}{64}\Gamma^3\right)\cos(v-v')\right],$$

and T_η is that part of T_0 which has e' as a factor when T_0 is developed in powers of e'. (T_{00} corresponds to the disturbing function used in obtaining the intermediate orbit.) The perturbations due to T_θ, T_α, T_η of the disturbing function are the θ-terms, the parallactic terms and the e'-terms, respectively.

4. The $\boldsymbol{\theta}$-terms. In this case $\delta_1 Q'$, $\delta_1 U'$, $\delta_1 v'$, and accordingly δQ, δU, etc., are of the form

$$\sum A \, {}^{\cos}_{\sin} (2F+2\xi j+il), \; j=0, \pm 1, \cdots, \pm 4; \; i=0, \pm 1, \cdots, \pm 8,$$

where the new argument $2F$ is defined as the value of $2v_0-2\theta_0$ without its periodic terms.

The multiplication of two series, one with the arguments $2\xi j+il$ and the other $2F+2\xi j+il$, is carried out by the help of a "product-table". In this table, for example, $(1,3)\times(-1,1)=-(0,4), (-2,-2)$ stands for

$$2\left(\frac{a}{2}\right) \sin(2\xi+3l) \times b \sin(2F-2\xi+l)$$
$$=\left(\frac{a}{2}\right) b\{-\cos(2F+4l)+\cos(2F-4\xi-2l)\}.$$

The difficulty of slow convergency in our successive approximations is avoided by adding unknown parameters to the coefficients of the series δQ, δU, and $\delta(v-v')$ after Brown-Brouwer. In the present work we have used 33 unknowns and obtained a set of linear equations for their determination. This difficulty is mainly due to the presence of small divisors, especially those connected with the argument $2F-2l$, combined with large values of some of the coefficients of δQ, δU, etc. in the equations of variations. We have carried out several approximations for reaching the desired accuracy $1\cdot 10^{-5}$ even by the use of the device referred to.

The complete equation for θ is

$$D\theta = \frac{\partial}{\partial \Gamma}(T_0+T_\theta),$$

so that
$$D\theta = U^{-4}[k_1(-1+\Gamma)+k_2(-1+\tfrac{1}{2}\Gamma)\cos(2v-2v')$$
$$+(1-\Gamma)\{k_1\cos(2v-2\theta)+k_2\cos(2v'-2\theta)\}$$
$$+\tfrac{1}{2}\Gamma k_2 \cos(2v+2v'-4\theta)].$$

It should be remembered that, in obtaining the intermediate orbit, we have neglected that part of the disturbing function which depends on θ, partly owing to the fact that $O(T_\theta)\sim O(\Gamma T_0)$ and $O(\Gamma)\sim 0.08$. In the equation for θ, however, the contribution of T_θ is of the same order as that of T_0 owing to the operation $\partial/\partial\Gamma$, and the preceding equation for $\delta\theta$ is not applicable.

In order to include higher orders of the variations, the equation for θ is completely integrated by the use of triple harmonic analysis: we compute the right-hand side of the equation for 486 special values with six values of $2F$, i.e. 0°, 180°, 60°, 300°, 120°, and 240°. After analysis and integration, the results are obtained in the form

$$\theta = \text{const.} + \theta_1 \text{v} + \sum \theta_{j,i} \sin(2\xi j+il) \qquad j=0,1,\cdots,4; \; i=0, \pm 1, \cdots, \pm 8,$$
$$+\sum \theta_{j,i,2} \sin(2F+2\xi j+il) \qquad j=0, \pm 1, \cdots, \pm 4;$$
$$+\sum \theta_{j,i,4} \sin(4F+2\xi j+il) \qquad i=0, \pm 1, \cdots, \pm 8,$$

The effects of the squares and the products of the variations are seen in the new series with the arguments $4F+2\xi j+il$, and also in the new values of the secular term and the terms with the arguments $2\xi j+il$, $2F+2\xi j+il$. These effects are: 0.002 in the secular term; 0.006 in the terms with the arguments $2\xi j+il$; 0.015 in the terms with the arguments $2F+2\xi j+il$; 0.010 in the terms with the arguments $4F+2\xi j+il$, where the unit of angles is radian.

5. The parallactic terms. These terms are of the form

$$\sum A {}^{\cos}_{\sin}\{(2j+1)\xi+il\}, \quad j=0,1,2,3;\ i=0, \pm 1, \cdots, \pm 8.$$

The multiplication of two series, one with the arguments $2\xi j+il$ and the other with $(2j+1)\xi+il$, is carried out effectively by the use of double harmonic analysis, the scheme of which is easy to prepare.

A remarkable contrast to the case of J-VIII is clearly manifested in the term with the argument $\xi-l$ in the series δ_Q. In the case of J-VIII, the smallness of this term has presented a curious feature. This has been due to the fact that a fairly large value of $\delta_1 Q'$ was nearly cancelled by the contribution due to $Q'_Q\delta_Q+Q'_U\delta_U+Q'_v\delta(v-v')$ in the right-hand side of the equation for δ_Q. But in the present case of J-IX the former is small, so that the latter remains large.

6. The e'-terms. The e'-terms are expressed in the form

$$\sum A {}^{\cos}_{\sin}(l'+2\xi j+il), \quad j=0, \pm 1, \cdots, \pm 4;\ i=0, \pm 1, \cdots, \pm 8,$$

where the new argument l' is defined as the mean motion of $n't_0-\varepsilon'+\varpi$. The "product-table" prepared for the computation of the θ-terms may be used for carrying out the multiplication of the series.

Another approach to the solution of the e'-terms has been attempted for trial. We transform the equations of variations into the corresponding matrix form, that is to say, we form a matrix from the coefficients of the terms of, say Q'_Q so that the multiplication $Q'_Q\times\delta_Q$ is carried out as if Q'_Q be a matrix and δ_Q a vector. Thus the computation of the successive approximations becomes easy to follow. In addition, the differential or the integral operators, D or D^{-1}, $[D^2+(1-\varpi_1^*)^2]^{-1}$, can be included in the coefficient-matrices; this also means that the effects of small divisors introduced by integration are included in the equations of variations themselves through their coefficients. Thus the transformation to the matrix form has the advantage in many devices in the numerical solution of the equations of variations. Although this transformation itself calls for a fairly lengthy work, it is worth-while for the computation of the θ-terms, the labour of which is greatly reduced by using our matrix method.

7. Comparison with observations: determination of the angular constants of integration. Nicholson has computed the jovicentric rectilinear coordinates of J-IX, together with the elliptic elements on the basis of his observations made in the intervals 1914–1918 and

1938–1943. In the present work we have carried out the comparison referring to the node and the inclination only, by using the observational data due to Nicholson. The accuracy aimed at in the present work being of one degree or 0.02 radian, our results are merely of a preliminary character. The detailed comparison should be made by referring to the space coordinates with the accuracy of the same order as the present theory, say 0.00001 jovicentric. In the present theory the motion of the node is expressed in a series of 204 terms, while we have used only 17 terms among them for our comparison.

The expression of the node or the inclination in terms of the true longitude contains four independent parameters, which depend on three constants of integration and one parameter defining a line fixed in the reference plane (the orbital plane of Jupiter) from which all angles are counted. It is noted that the observational values of the node and the inclination are referred to the ecliptic. Then we have framed the following scheme: to transform the independent variables from the true longitude to the time; to transform the reference frames from the ecliptic system to that referred to the orbital plane of Jupiter; to establish the relation between the four parameters and the three constants of integration; to determine the numerical values of the parameters so that the theoretical values of the node are consistent with observations; to determine the numerical values of the integration constants; and finally to compute the inclination with the values of the parameters determined in this way.

Thus we have determined the values of the four parameters, which enable us to reduce the values of "$O-C$" to 1.8 degrees at most throughout the intervals 1914–1918 and 1930–1943 for both the node and the inclination. In addition, these values of the parameters are found to be fairly consistent with the observations:

	the present theory	observation
the constant term of the node (epoch: 1942)	67°.0	70°
the constant term of the longitude of the perijove (epoch: 1942)	319°.8	319°.

The difference between the two values shown in the first line may be due to the fact that Nicholson did not take into account the long-period term of 44 year period in the procedure of separating the constant term.

8. Discussion. From the preceding results it seems probable that the values of "$O-C$" in the node or the inclination are much reduced if we use a larger number of terms in the series of the node, the inclination and the time. Also we see that the values of the adopted constants which are deduced by Nicholson from his observations and with which we started the whole computation of general perturbations are correct in their accuracy. It is remarked, however, that the value of the mean inclination $22°28'$ replaces the value shown in the preceding paper.[1]

Now we see how important the character of the θ-terms especially in the perturbations of the node and the inclinations. Without the θ-terms, the theoretical value of the mean motion of the node $3°.80$/year is too small compared with the observational values $4°.44$/year and their difference amounts to $19°$ for an interval of 30 years. This discrepancy is beyond remedy. On the other hand the effects of the second order variations in the θ-terms are:

in the mean motion of the node $3°.80/\text{year} \to 4°.19/\text{year}$;

in the amplitude of the long-period term with the period 44 years $6°.7 \to 5°.9$.

It is noticed that the difference of $19°$ is not remedied by the term with amplitude $6°.7$, while the difference of $7°.5$ can be nearly disposed of by the term with amplitude $5°.9$. These effects are considerable for the motion of the node. We are in a similar situation in the case of the inclination. In order to see the actual situation, we need observations of recent years. Considering such important effects of the θ-terms, it seems necessary to adopt a certain device for including parts of T_θ into the intermediate orbit, without adding serious complication.

In conclusion the writer wishes to express his hearty thanks to Professor Hagihara for his suggesting this problem and for his encouragement, also to Professor Hirose for his valuable advice and encouragement. The writer is also indebted to Professor Kaburaki for his constant encouragement.

References

1) G. Hori: Proc. Japan Acad., **33**, 392 (1957); Publ. A. S. Japan, **9**, 51 (1957).
2) E. W. Brown and D. Brouwer: Trans. Yale Univ. Obs., **6**, no. 8 (1932).
3) S. B. Nicholson: Ap. J., **100**, 57 (1944).