83. *Representation of Some Topological Algebras. I*

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1. Introduction. In the study of the algebra of all continuous endomorphisms on a locally convex Hausdorff topological vector space, the abundance of endomorphisms of finite rank plays an important role. But, for an arbitrary algebra, there is of course no such a convenience in general, and it is interesting to determine an algebra which can be embedded into the algebra of all continuous endomorphisms on a locally convex vector space in such a manner that the embedded algebra contains every continuous endomorphism of finite rank. In this paper, we deal with this problem.

We shall be exclusively concerned with algebras over the complex or the real number field. A *topological algebra* is by definition an algebra and topological vector space such that the ring multiplication is separately continuous. Let E be an algebra; a topology with which E is a topological algebra is said to be *compatible* with the structure of E. As can readily be seen, we obtain the following proposition:

In order that a filter base $\mathfrak{B}$ on an algebra E is a fundamental system of neighbourhoods of 0 in E for a topology compatible with the structure of E, it is necessary and sufficient that $\mathfrak{B}$ possesses the following properties:[1]

1° For any number $\lambda \neq 0$, and for any $V \in \mathfrak{B}$, λV belongs to $\mathfrak{B}$.

2° Every member V of $\mathfrak{B}$ is absorbing, i.e. for any $x \in E$, there exists a number $\lambda \neq 0$ such that $\lambda x \in V$.

3° For any $U \in \mathfrak{B}$, there exists $V \in \mathfrak{B}$ such that $V+V \subseteq U$.

4° If $x \in E$, then for any $U \in \mathfrak{B}$, there exists $V \in \mathfrak{B}$ such that $xV \subseteq U$ and $Vx \subseteq U$.

Notice that we may assume $\mathfrak{B}$ to consist of circled sets.[2]

A topological algebra which is at the same time a locally convex topological vector space is called a *locally convex algebra*.

2. Bounded sets. Let E be a topological algebra. A subset A of E is called *left bounded* if for any neighbourhood V of 0 in E there exists a neighbourhood U of 0 in E such that $UA \subseteq V$. In an analogous way, we define right boundedness. A subset of E which

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1) We employ the following notations: $\lambda A = \{\lambda x;\ x \in A\}$, $aA = \{ax;\ x \in A\}$, $AB = \{xy;\ x \in A,\ y \in B\}$, $A+B = \{x+y;\ x \in A,\ y \in B\}$.

2) A subset A of a vector space is said to be *circled* if $x \in A$ and $|\lambda| \leqq 1$ imply $\lambda x \in A$.

is both left bounded and right bounded is termed *hyperbounded*. We say that a subset A of E is *bounded* whenever it is bounded as a subset of the topological vector space E, that is, if, for any neighbourhood V of 0 in E there exists a number $\lambda \neq 0$ such that $\lambda A \subseteq V$.

The proofs of the following propositions are all straightforward:

(1) Every subset of a left (right) bounded set is left (right) bounded, and so the intersection of an arbitrary non-void family of left (right) bounded sets is left (right) bounded.

(2) The union of a finite number of left (right) bounded sets is left (right) bounded.

(3) If A is left (right) bounded, then so is λA for any number λ.

(4) If A and B are both left (right) bounded, then AB and $A+B$ are both left (right) bounded.

(5) The closure of a left (right) bounded set is left (right) bounded.

(6) If A is left bounded and if B is bounded, then BA is bounded.

(7) If A is right bounded and if B is bounded, then AB is bounded.

(8) If E is locally convex, then the convex hull of a left (right) bounded set is left (right) bounded.

(9) If E has an identity, then every left (right) bounded set is bounded.

The property 4° of the fundamental system of neighbourhoods of 0 shows that every finite set in E is hyperbounded. Therefore, from (4) it follows that, for any $x \in E$, the sets xA, Ax and $x+A$ are left (right) bounded if A is left (right) bounded. By (6) and (7), we see further that if A is bounded, then the sets xA and Ax are both bounded for any $x \in E$.

Let A and V be two subsets of E. We shall denote by $W_l(A, V)$ the set of all elements $x \in E$ such that $xA \subseteq V$, and by $W_r(A, V)$ the set of all elements $x \in E$ such that $Ax \subseteq V$.

THEOREM 1. *Let E be a topological algebra. If E is a t-space,*[3] *then every bounded set in E is hyperbounded.*

Proof. Let A be a bounded set in E. To prove that A is left bounded, it is sufficient to show that the set $W_l(A, V)$ is a neighbourhood of 0 in E for any closed convex and circled neighbourhood V of 0 in E. Now let x be an arbitrary element of E, then there exists a number $\lambda \neq 0$ such that $\lambda xA \subseteq V$, as we have pointed out above. Hence the set $W_l(A, V)$ is absorbing. Suppose that x does not belong to $W_l(A, V)$, then we can find an element a of A such that $xa \bar{\in} V$, and

3) In French "espace tonnelé". Cf. N. Bourbaki: Espaces Vectoriels Topologiques, Chaps. III–V, Hermann, Paris (1955).

so we have $(xa+U)\frown V=\phi$ for some neighbourhood U of 0 in E. Take a neighbourhood W of 0 in E for which we have $Wa\subseteq U$. Then, for any $y\in W$, the set $(x+y)A$ is not contained in V, since $(x+W)a$ and V do not meet. In other words, $x+y$ does not belong to $W_l(A, U)$, proving that $W_l(A, V)$ is closed in E. Since $W_l(A, V)$ is convex and circled, this is a neighbourhood of 0 in E. Similarly, we can prove that the set $W_r(A, V)$ is a neighbourhood of 0 in E for each neighbourhood V of 0 in E.

A topological algebra E is said to be *normable* if the topology of E can be defined by means of a norm with which E is a normed algebra.

THEOREM 2. *Let E be a locally convex Hausdorff topological algebra. Then E is normable if and only if there exists a neighbourhood of 0 in E which is left (or right) bounded and bounded.*

Proof. The only if part is obvious. Let U be the convex circled neighbourhood of 0 which is left bounded and bounded. Then, there exists a neighbourhood V of 0 such that $VU\subseteq U$. Since U is bounded, we have $\lambda U\subseteq V$ for some number $\lambda\neq 0$, and so $\lambda U\lambda U\subseteq\lambda U$.

Let S be a set, E a topological algebra, and $\mathfrak{S}$ a family of subsets of S. The set $\mathcal{F}(S, E)$ consisting of all mappings of S into E forms an algebra with pointwise definition of the algebraic operations.

THEOREM 3. *With the notion given above, if H is a subalgebra of $\mathcal{F}(S, E)$, then in order that the topology of uniform convergence on members of $\mathfrak{S}$ is compatible with the structure of H, it is sufficient that, for any $u\in H$, the image of each member of $\mathfrak{S}$ under u is hyperbounded and bounded.*

Proof. It will suffice to show that the ring multiplication in H is separately continuous. Let u be an element of H, and A an arbitrary member of $\mathfrak{S}$. Since $u(A)$ is hyperbounded, we can find, for each neighbourhood V of 0 in E, a neighbourhood U of 0 in E such that $u(A)U\subseteq V$ and $Uu(A)\subseteq V$. But then, we have $uW(A, U)\subseteq W(A, V)$ and $W(A, U)u\subseteq W(A, V)$, where $W(A, U)$ denotes the set of all $v\in H$ such that $v(x)\in U$ for any $x\in A$.

Let E be a topological algebra. Let us denote by $\mathfrak{L}(\mathfrak{R})$ the family of all left (right) bounded sets in E, and by $\mathfrak{A}(\mathfrak{B})$ a subfamily of $\mathfrak{L}(\mathfrak{R})$ such that $xA\in\mathfrak{A}(Ax\in\mathfrak{B})$ for any $A\in\mathfrak{A}(\in\mathfrak{B})$ and any $x\in E$. Then, the family $\mathfrak{V}$ of all sets of the form $W_l(A, V)$, where $A\in\mathfrak{A}$ and V runs through the fundamental system of neighbourhoods of 0 in E, is a fundamental system of neighbourhoods of 0 for a topology compatible with the structure of E. In fact, the family $\mathfrak{V}$ satisfies the conditions 1°–4°. For example, 2° follows from the fact that xA is bounded for any $A\in\mathfrak{A}$, and 4° is apparent since we have, for any $W_l(A, U)\in\mathfrak{V}$,

$$W_l(xA, U)x\subseteq W_l(A, U) \quad\text{and}\quad xW_l(A, V)\subseteq W_l(A, U),$$

by taking a neighbourhood V of 0 in E such that $xV \subseteq U$. We shall call this topology the *left* $\mathfrak{A}$*-topology*. In an analogous way, we define the *right* $\mathfrak{B}$*-topology*. Obviously, these topologies are coarser than the original one of E. But if E has an identity and if it belongs to a member of $\mathfrak{A}(\mathfrak{B})$, then the left $\mathfrak{A}$-topology (right $\mathfrak{B}$-topology) is identical with the original topology of E. If the algebra is locally convex, then the family $\mathfrak{A}$ as well as $\mathfrak{B}$ can be always supposed to consist of closed, convex and circled sets, and to be such that the closed convex hull of the union of any finite number of the members of $\mathfrak{A}$ belongs to $\mathfrak{A}$, and that of $\mathfrak{B}$ to $\mathfrak{B}$.

THEOREM 4. *With the notion given above, every left bounded set in E is left bounded for any right $\mathfrak{B}$-topology; if $\mathfrak{B}'$ is a subfamily of $\mathfrak{R}$ such that $BA \in \mathfrak{B}$ for any $A \in \mathfrak{B}'$ and any $B \in \mathfrak{B}$, then every member of $\mathfrak{B}'$ is right bounded for the right $\mathfrak{B}$-topology.*

Notice that, similar results hold for the left $\mathfrak{A}$-topology.

Proof. Let A be a left bounded set in E. Then, for any neighbourhood U of 0 in E, there exists a neighbourhood V of 0 in E such that $VA \subseteq U$. Hence, for each $B \in \mathfrak{B}$, the set $W_r(B, V)A$ is contained in $W_r(B, U)$, proving the first assertion. If $A \in \mathfrak{B}'$, then we have $AW_r(BA, U) \subseteq W_r(B, U)$ for any neighbourhood U of 0 in E, and any $B \in \mathfrak{B}$. Since $BA \in \mathfrak{B}$, A is right bounded for the right $\mathfrak{B}$-topology.

Denote by $E_{\mathfrak{B}}$ the algebra E with the right $\mathfrak{B}$-topology. Since every $A \in \mathfrak{A}$ is also left bounded in $E_{\mathfrak{B}}$, we can consider *in* $E_{\mathfrak{B}}$ the left $\mathfrak{A}$-topology, which we shall call the $(\mathfrak{A}, \mathfrak{B})$*-topology*.

3. The representation theorems. Let X be a topological vector space. We denote by $\mathcal{L}(X, X)$ the set of all continuous linear mappings of X into itself. This is itself not only a vector space, but also an algebra by defining the ring multiplication of two elements u, v of $\mathcal{L}(X, X)$ as follows: $uv(x) = u(v(x))$ for each $x \in X$. A linear mapping of X into itself is said to be of *finite rank* if its range is a finite dimensional vector subspace of X.

Let E be an algebra satisfying the following two conditions:

(i) There exists a non-zero element $a \in E$ such that, for any $u \in E$ we can find a number λ for which we have $aua = \lambda a$.

(ii) For any non-zero elements u, v of E, there exists an element $w \in E$ such that $uwv \neq 0$.

Then, it is not hard to prove that

(i′) There exists a non-zero idempotent $p \in E$ such that, for any element $u \in E$, we can find a number λ for which we have $pup = \lambda p$. In fact, the conditions (i) and (ii) ensure the existence of an element $v \in E$ such that $ava = a$. Then $p = av$ as well as $p = va$ possesses the required property, as is easily checked.

For any $x = up \in Ep$ and any $y = pv \in pE$, there exists a unique number λ such that $yx = pvup = \lambda p$; we put then $\langle x, y \rangle = \lambda$. It is easy to see that the mapping $(x, y) \to \langle x, y\rangle$ is a bilinear functional over $Ep \times pE$. Now, with respect to the bilinear functional $\langle x, y\rangle$, the vector spaces Ep and pE constitute a *separated* dual system, that is, by definition the following conditions are satisfied:

(a) For any non-zero $x \in Ep$, there exists $y \in pE$ such that $\langle x, y\rangle \neq 0$;

(b) For any non-zero $y \in pE$, there exists $x \in Ep$ such that $\langle x, y\rangle \neq 0$.

For, if $x = up \neq 0$, then we can find an element $w \in E$ such that $pwup \neq 0$; and if $y = pv \neq 0$, then we have also $pvwp \neq 0$ for some $w \in E$.

Let us denote by X the vector space Ep with the weak topology $\sigma(Ep, pE)$,[4] then X is a locally convex Hausdorff vector space whose dual is pE. Let $u \in E$; and put $\tilde{u}(x) = ux$ for every $x \in X$. It is clear that $\tilde{u}$ is a linear mapping of X into itself. On the other hand, since $yu \in pE$ for any $y \in pE$, the mapping $\tilde{u}$ is weakly continuous, and thus $\tilde{u}$ belongs to $\mathcal{L}(X, X)$. As can readily be seen, the mapping $u \to \tilde{u}$ of the algebra E onto the subalgebra $\tilde{E} = \{\tilde{u}; u \in E\}$ of $\mathcal{L}(X, X)$ is a homomorphism. But this is indeed an isomorphism, since, for any distinct elements u, v of E, there exists an element $w \in E$ such that $(u-v)wp \neq 0$. Now, let $up \in X$ and let $pv \in pE$, then for each $x = wp \in X$, we have

$$\langle x, pv\rangle up = \langle wp, pv\rangle up = u\langle wp, pv\rangle p = upvwp = \widetilde{upv}(x).$$

This proves that $\tilde{E}$ contains every mapping of the form $x \to \langle x, pv\rangle up$, and so every continuous linear mapping of finite rank. We have thus obtained the following

THEOREM 5. *If E is an algebra satisfying the conditions* (i) *and* (ii), *then there exists a locally convex Hausdorff vector space X such that E is isomorphic* (*algebraically*) *with a subalgebra of $\mathcal{L}(X, X)$ containing all continuous linear mappings of finite rank.*

The following theorem is now evident.

THEOREM 6. *If E is an algebra satisfying the condition* (i′), *then there exists a locally convex vector space X such that E is homomorphic* (*algebraically*) *with a subalgebra of $\mathcal{L}(X, X)$ containing all continuous linear mappings of finite rank.*

In the above consideration, let us suppose further that E is a Hausdorff topological algebra. Then there exists a neighbourhood U of 0 in E such that $\lambda p \in U$ when and only when $|\lambda| \leqq 1$. Let $y_1, y_2, \cdots, y_m$ be a finite set of points of pE, then we can find a neighbourhood V of 0 in E for which we have $y_i V \subseteq U$ $(i=1, 2, \cdots, m)$. Hence

$$\begin{aligned}(y_1, y_2, \cdots, y_m)^\circ &= \{x \in X;\ |\langle x, y_i\rangle| \leqq 1 \text{ for } i=1, 2, \cdots, m\} \\ &= \{x \in Ep;\ y_i x \in U \text{ for } i=1, 2, \cdots, m\} \supseteq V \frown Ep.\end{aligned}$$

It follows that, if $x_1, x_2, \cdots, x_n \in X$, then the image of the set $W_l(\{x_1, x_2, \cdots, x_n\}, V)$ under the mapping $u \to \tilde{u}$ is contained in the set $W(\{x_1, x_2, \cdots, x_n\}, (y_1, y_2, \cdots, y_m)^\circ)$.[5] Since the set $W_l(\{x_1, x_2, \cdots, x_n\}, V)$ is a neigh-

4) For this notation, see N. Bourbaki: Loc. cit.

5) Let A and V be two subsets of a topological vector space X. Then we denote by $W(A, V)$ the set of all $u \in \mathcal{L}(X, X)$ such that $u(A) \subseteq V$.

bourhood of 0 in E, the mapping $u \to \tilde{u}$ of E into $\mathcal{L}_s(X, X)$ is continuous, where $\mathcal{L}_s(X, X)$ is the algebra $\mathcal{L}(X, X)$ with the topology of pointwise convergence. Therefore, we can state as follows:

THEOREM 7. *Let E be a Hausdorff topological algebra satisfying the conditions* (i) *and* (ii), *then there exists a locally convex Hausdorff vector space X such that E is mapped, by a continuous isomorphism, onto a subalgebra of $\mathcal{L}_s(X, X)$ containing all continuous linear mappings of finite rank.*

Moreover, suppose that the algebra E is locally convex and satisfies the following conditions:

(iii) Every right bounded set in pE is relatively compact for the left $\{\{x\}; x \in Ep\}$-topology.

(iv) Let $\mathfrak{A}(\mathfrak{B})$ be the family of all left (right) bounded sets contained in $Ep(pE)$, then the $(\mathfrak{A}, \mathfrak{B})$-topology is identical with the original topology of E.

Under these assumptions, we will show that the isomorphism $u \to \tilde{u}$ can be a homeomorphism. Let us denote again by X the vector space Ep with the right $\mathfrak{B}$-topology. Since $Bu \in \mathfrak{B}$ for any $B \in \mathfrak{B}$ and any $u \in E$, we see that $\tilde{E}$ is contained in the algebra $\mathcal{L}(X, X)$. On the other hand, since E is a Hausdorff space, we can find a neighbourhood U of 0 in E such that $\lambda p \in U$ when and only when $|\lambda| \leqq 1$. We have then, for each $B \in \mathfrak{B}$,

$$B^\circ = \{x \in X;\ |\langle x, y\rangle| \leqq 1 \text{ for every } y \in B\} = W_r(B, U) \frown Ep,$$

and hence, by the condition (iii), the dual space of X is pE. This ensures that the algebra $\tilde{E}$ contains all continuous linear mappings of finite rank. Now let $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ and let V be a neighbourhood of 0 in E. Then the image of the set $W_l(A, W_r(B, V))$ under the mapping $u \to \tilde{u}$ is $W(A, W_r(B, V) \frown Ep) \frown \tilde{E}$. Therefore, the isomorphism $u \to \tilde{u}$ is a homeomorphism of E onto the subalgebra $\tilde{E}$ of $\mathcal{L}_{\mathfrak{A}}(X, X)$, where $\mathcal{L}_{\mathfrak{A}}(X, X)$ is the algebra $\mathcal{L}(X, X)$ with the topology of uniform convergence on the members of $\mathfrak{A}$. To prove that this topology is compatible with the structure of $\mathcal{L}(X, X)$, it will suffice to show that each member A of $\mathfrak{A}$ is bounded in X. For any $u \in E$, the set uB is bounded in E, and so A is bounded for the weak topology $\sigma(X, pE)$. Therefore A is bounded in X. We can conclude thus as follows:

THEOREM 8. *Let E be a locally convex Hausdorff topological algebra satisfying the conditions* (i), (ii), (iii) *and* (iv). *Then there exists a locally convex Hausdorff vector space X such that E can be identified with a subalgebra of $\mathcal{L}_{\mathfrak{S}}(X, X)$ containing all continuous linear mappings of finite rank, where $\mathfrak{S}$ is a family of bounded sets in X.*

Let E be a Hausdorff topological commutative algebra. If E satisfies the conditions (i) and (ii), then E is isomorphic (algebraically and topologically) with the scalar field. In fact, the vector spaces Ep and pE are both of dimension 1, and hence the dimension of E is also 1. Therefore E is isomorphic with the scalar field.