78. On the Differentiability of Semi-groups of Linear Operators

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§ 1. The purpose of the present note is to prove the two theorems given below which may be applied to the "abstract Cauchy problem" connected with the semi-group theory.[1] We recall, for the sake of exposition, some of the basic results from the theory. Let a one-parameter family T_t, $0 \leq t < \infty$, of bounded linear operators on a complex Banach space X into X satisfy the semi-group conditions:

(1.1) $\qquad T_t T_s = T_{t+s}$, $T_0 = I$ (the identity operator),

(1.2) $\qquad s - \lim_{t \to t_0} T_t x = T_{t_0} x$, $t_0 \geq 0$, $x \in X$,

(1.3)[2] $\qquad \| T_t \| \leq 1$, $t \geq 0$.

Then the infinitesimal generator A of T_t defined by

(1.4) $\qquad Ax = s - \lim_{h \downarrow 0} h^{-1}(T_h - I)x$

is a closed linear operator with its domain $D(A)$ dense in X such that

(1.5) $\qquad R(\lambda, A)x = (\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} T_t x \, dt, \ Re(\lambda) > 0$

exists as a bounded linear operator on X into X satisfying

(1.6) $\qquad \| R(\lambda, A) \| \leq (Re(\lambda))^{-1}$, $Re(\lambda) > 0$.

Conversely, let a linear operator A with its domain $D(A)$ dense in X and taking the values in X admit the resolvent $(\lambda I - A)^{-1} = R(\lambda, A)$ which satisfies (1.6). Then A is the infinitesimal generator of a semi-group T_t which enjoys (1.1)–(1.5).

We have the representations

(1.7) $\qquad T_t x = s - \lim_{n \to \infty} (I - n^{-1} tA)^{-n} x$, $x \in X$,

(1.7)' $\qquad T_t x = s - \lim_{n \to \infty} \exp(tA(I - n^{-1}A)^{-1})x$, $x \in X$,

uniformly in t in any compact interval of t. We have moreover,

(1.8) $\qquad T_t x = s - \lim_{|\tau| \uparrow \infty} (2\pi)^{-1} \int_{\sigma - i\tau}^{\sigma + i\tau} e^{\lambda t} R(\lambda, A)x \, d\lambda$, $x \in D(A)$, $t > 0$, $\sigma = Re(\lambda) > 0$.

In these senses, we may write $T_t = \exp(tA)$.

(1.1), (1.2) and (1.4) imply that $T_t = \exp(tA)$ solves, for given A,

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1) E. Hille and R. S. Phillips: Functional Analysis and Semi-groups, New York (1957). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, **1**, 15–21 (1948).

2) (1.1) and (1.2) imply that $\lim_{t \to \infty} t \log \| T_t \| = \beta < \infty$. Thus, by taking $e^{-\beta t} T_t$ in place of the original T_t, we may assume that the condition (1.3) is satisfied.

"the abstract Cauchy problem" (ACP):

(1.9)   for $x \in D(A)$ and $t \geq 0$, we have

$$T_t'x = s - \lim_{h \to 0} h^{-1}(T_{t+h} - T_t)x = AT_tx = T_tAx, \quad s - \lim_{t \downarrow 0} T_tx = x.$$

However, there are semi-groups satisfying the condition:

(1.10)   for every $x \in X$ and every $t > 0$, we have $T_tx \in D(A)$.

In such case, T_t solves the $(ACP)_0$:

(1.9)′   for every $x \in X$ and every $t > 0$, we have

$$T_t'x = AT_tx, \quad s - \lim_{t \downarrow 0} T_tx = x.$$

There are also semi-groups T_t which can be extended to T_λ which are analytic in a sector of the complex λ-plane of the form

(1.11) $|\operatorname{Arg} \lambda| < \alpha < \pi/2$.

(1.10)   implies that $T_t' = AT_t$ is, as a closed linear operator defined on the Banach space X, a bounded linear operator and hence we have, by (1.1) and (1.10),

$$T_t'' = (T_t')' = (T_{t-s}AT_s)' = (AT_{t/2})^2 = (T_{t/2}')^2$$

or more generally

(1.12) $T_t^{(n)} = (AT_{t/n})^n = (T_{t/n}')^n$.

E. Hille and R. S. Phillips have discussed such differentiability properties of semi-groups and obtained conditions upon the behaviour of the resolvent $R(\lambda, A)$ which imply the differentiability for $t > 0$ of $T_t = \exp(tA)$. The following two theorems are suggested by their researches and may be applied to discuss the behaviour near $t = 0$ of solutions $T_tx = \exp(tA)x$ of $(ACP)_0$.

   Theorem 1. T_t' exists for $t > 0$ and

(1.13) $\overline{\lim_{t \downarrow 0}} \, t \, \|T_t'\| < \infty$

if and only if

(1.14) $\overline{\lim_{|\tau| \uparrow \infty}} \, |\tau| \cdot \|R(1 + i\tau, A)\| < \infty$.

Moreover, the semi-group T_t of this class can be extended to T_λ which is analytic in a sector of the form

(1.11)′ $|\lambda - t| < t/eC$, *where* $C = \sup_{t > 0} \|(e^{-t/2}T_t)'\|$.

   Theorem 2. T_t' exists for $t > 0$ and

(1.13)′ $\overline{\lim_{t \downarrow 0}} \, t \cdot \log \|T_t'\| = 0$

if and only if

(1.14)′ $\overline{\lim_{|\tau| \uparrow \infty}} \, \log |\tau| \cdot \|R(1 + i\tau, A)\| = 0$.

   Remark. The semi-groups T_t are assumed to satisfy (1.1)–(1.3) so that the theorems are exhibited under the assumption (1.6).

   §2. *Proof of Theorem 1.* Let $\overline{\lim_{t \downarrow 0}} \, t \, \|T_t'\| < \infty$. Then, since $\|T_t\| \leq 1$, we have $\sup_{t > 0} t \, \|(e^{-t/2}T_t)'\| = C < \infty$. Hence, by applying (1.12) to the semi-group $S_t = e^{-t/2}T_t$, we see that

$$(t/n)^n \, \| S_t^{(n)} \| = (t/n)^n \, \| (S'_{t/n})^n \| \leqq C^n \quad \text{for } t>0, \ n\geqq 1.$$

Thus

(2.1) $$(n!)^{-1} |\lambda - t|^n \cdot \| S_t^{(n)} \| \leqq (n!)^{-1} (nCt^{-1}|\lambda - t|)^n,$$

and the analytical continuation

(2.2) $$S_\lambda = e^{-\lambda/2} T_\lambda = e^{-1/2} T_t + \sum_{n=1}^\infty (n!)^{-1}(\lambda - t)^n (e^{-t/2} T_t)^{(n)}$$

is possible for λ satisfying (1.11)$'$. Moreover, we have, from (2.1)–(2.2),

(2.3) $$\| S_\lambda \| = \| e^{-\lambda/2} T_\lambda \| \leqq (1 - eCt^{-1}|\lambda - t|)^{-1} \quad \text{when } eCt^{-1}|\lambda - t| < 1.$$

Let $\tau > 0$. Then, because of the analyticity of S_λ in λ, we can deform the path of integration $0 \leqq t < \infty$ of the integral

$$R(1+i\tau, A) = \int_0^\infty e^{-(1+i\tau)t} T_t dt = \int_0^\infty e^{-i\tau t} e^{-t/2} S_t dt$$

to the path $te^{-i\varphi}$, $0 \leqq t < \infty$, with a fixed φ such that $0 < \tan \varphi < 1/eC$:

$$R(1+i\tau, A) = \int_0^\infty \exp(-i\tau t e^{-i\varphi}) \exp(-te^{-i\varphi}/2) S_{te^{-i\varphi}} e^{-i\varphi} dt.$$

Such deformation is possible thanks to the convergence factor $\exp(-te^{-i\varphi}/2)$ and the estimate (2.3). Thus it is easy to obtain (1.14).

Let conversely $\| R(1+i\tau, A) \| \leqq C/|\tau|$ for large $|\tau|$. Then, by virtue of the resolvent equation

(2.4) $$R(\lambda, A) = \sum_{n=0}^\infty R(1+i\tau, A)^{n+1}(1+i\tau - \lambda)^n$$

valid for $|1+i\tau - \lambda| \cdot \| R(1+i\tau, A) \| < 1$, we see that $R(\lambda, A)$ exists and is analytic in λ for large value of $|\tau| = |Im(\lambda)|$ outside a sector of the left half λ-plane defined by the boundary curve $\lambda(s) = \sigma(s) + i\tau(s)$ which satisfies

$$\lim_{\tau(s)\uparrow\infty} (-\sigma(s)/\tau(s)) = \tan \varepsilon = \lim_{\tau(s)\downarrow -\infty} \sigma(s)/\tau(s), \quad \varepsilon > 0.$$

Moreover, (2.4) shows that $R(\lambda, A)$ is of order $|\tau^{-1}|$ when $|\tau| = |Im(\lambda)|$ tends to ∞ outside the above sector and lying in the left half λ-plane. Therefore, by deforming the path of integration $-\infty < \tau < \infty$ of the integral

$$T_t x = (2\pi)^{-1} i \int_{-\infty}^\infty e^{(1+i\tau)t} R(1+i\tau, A) x \, d\tau, \ x \in D(A), \ t>0,$$

to the path $\tilde{\lambda}(s) = 2^{-1}\sigma(s) + i\tau(s)$, we see that $T'_t x$ exists for every $x \in X$ and every $t > 0$. Hence we easily see that T'_t satisfies (1.13).

Remark. The above proof shows that T_t is differentiable even at $t=0$ if T'_t exists for $t>0$ and $\varlimsup_{t\downarrow 0} t \| T'_t \| < e^{-1}$. This remarkable result was proved by E. Hille[3] in another way.

    3) E. Hille: On the differentiability of semi-group operators,   Acta Sci. Math. Szeged, **12**, 19–24 (1950).

Proof of Theorem 2.[4]   We have, by partially integrating (1.5) and making use of the existence of T_t' for $t>0$,

$$R(1+i\tau, A) = \int_0^\delta e^{-(1+i\tau)t} T_t dt + (1+i\tau)^{-1} e^{-(1+i\tau)\delta} T_\delta$$
$$+ (1+i\tau)^{-1} \int_\delta^\infty e^{-(1+i\tau)t} T_t' dt.$$

$\|T_t'\|$ is, by $T_t' = AT_t$, (1.1) and (1.2), monotone decreasing in t. Thus

(2.5) $\|R(1+i\tau, A)\| \leq \delta + (1+\tau^2)^{-1/2}(1+\|T_\delta'\|)$.

Now $t^{-1}\|T_t'\|$ is monotone decreasing in t with $\|T_t'\|$ and so there are two cases: $\lim_{t\downarrow 0} t^{-1}\|T_t'\| = \infty$ and $\lim_{t\uparrow 0} t^{-1}\|T_t'\| < \infty$. In the latter case, we have, by Theorem 1, $\lim_{|\tau|\uparrow\infty} \tau \cdot \|R(1+i\tau, A)\| < \infty$ and hence (1.14)$'$. In the former case, we take $|\tau|$ as equal to $\delta^{-1}\|T_\delta'\|$. Then, since $\lim_{t\downarrow 0} t \cdot \log \|T_t'\| = 0$ by the assumption, we have $\lim_{\delta\downarrow 0} \delta (\log \delta + \log \tau) = 0$, viz. $\lim_{\delta\downarrow 0} \delta \log \tau = 0$. This proves (1.14)$'$ by (2.5).

Let, conversely, (1.14)$'$ hold good. By virtue of the resolvent equation (2.4) and the assumption (1.14)$'$, we see that $R(\lambda, A)$ is analytic in λ for large value of $|\tau| = Im(\lambda)$ outside a curved sector of the left half λ-plane with boundary curve $\lambda(s) = \sigma(s) + i\tau(s)$ which satisfies the condition

$$|\sigma(s)| = \varepsilon(s)^{-1} \log |\tau(s)|, \quad 0 < \varepsilon(s) \quad \text{and} \quad \lim_{|s|\uparrow\infty} \varepsilon(s) = 0,$$
$$\lim_{s\uparrow\infty} \tau(s) = \infty, \quad \lim_{s\downarrow-\infty} \tau(s) = -\infty.$$

Moreover, by (2.4), $\|R(\lambda, A)\|$ is of order $o(1/\log |\tau|)$ when $|\tau| = |Im(\lambda)|$ tends to ∞ outside the above sector and lying in the left half λ-plane. Thus, by deforming the path of integration $-\infty < \tau < \infty$ of the integral

$$T_t x = (2\pi)^{-1} i \int_{-\infty}^\infty e^{-(1+i\tau)t} R(1+i\tau, A) x d\tau, \quad x \in D(A), t>0,$$

to the path $\tilde{\lambda}(s) = 2^{-1}\sigma(s) + i\tau(s)$, we see that $T_t' x$ exists for every $x \in X$ and every $t>0$. Hence we easily see that (1.13)$'$ holds good.

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    4) The proof of the "only if part" is suggested by E. Hille, loc. cit. in 3). That the condition (1.14)$'$ implies the existence of T_t' for $t>0$ has already been proved in Hille and Phillips, loc. cit. in 1).