76. *On Tangent Bundles of Order 2 and Affine Connections*

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In this paper, the author will show that the classical connections, for instance, the affine, projective, conformal connections, can be considered from a unificative standpoint by means of the concept of tangent bundles of order 2, although they can be also discussed through the theory of connections of vector bundles.[1] We shall investigate the relations between this theory and the ones of C. Ehresmann and S. S. Chern[2] in Mathematical Journal of Okayama University, 8.

1. **The group $\mathfrak{L}_n^2$.** According to C. Ehresmann,[3] let L_n^2 be the group of the infinitesimal isotropies of order 2 at the origin of R^n, whose any element is written as a set of numbers (a_i^j, a_{ik}^j) such that $|a_i^j| \neq 0$, $a_{ik}^j = a_{ki}^j$. We can easily see that the set $\mathfrak{L}_n^2$ of (a_i^j, a_{ik}^j) such that only $|a_i^j| \neq 0$, also forms a group containing L_n^2 as a subgroup with the multiplication as follows:

For any two $\alpha, \beta \in \mathfrak{L}_n^2$, $\gamma = \alpha\beta$ is defined by

$$a_i^j(\gamma) = a_k^j(\alpha)\, a_i^k(\beta), \tag{1.1}$$

$$a_{ik}^j(\gamma) = a_h^j(\alpha)\, a_{ik}^h(\beta) + a_{hl}^j(\alpha)\, a_i^h(\beta)\, a_k^l(\beta). \tag{1.2}$$

By (1.1), we have a natural homomorphism σ: $\mathfrak{L}_n^2 \to L_n^1 = GL(n, R)$ such that

$$a_i^j(\sigma(\alpha)) = a_i^j(\alpha). \tag{1.3}$$

As is well known, we may consider L_n^1 as a subgroup of L_n^2, regarding the second coordinates a_{ik}^j of their elements as zero. Let $\mathfrak{N}_n^2$ be the kernel of σ. By means of (1.2), for any $\alpha, \beta \in \mathfrak{N}_n^2$, we have

$$a_{ik}^j(\alpha\beta) = a_{ik}^j(\alpha) + a_{ik}^j(\beta),$$

hence $\mathfrak{N}_n^2$ is a vector space of dimension n^3. We define a mapping $\eta: \mathfrak{L}_n^2 \to \mathfrak{N}_n^2$ by

$$\eta(\alpha) = \sigma(\alpha^{-1})\alpha. \tag{1.4}$$

Then, we can write uniquely any element α of $\mathfrak{L}_n^2$ as a product of $\sigma(\alpha) \in L_n^1$ and $\eta(\alpha) \in \mathfrak{N}_n^2$

1) See T. Ōtsuki: Geometries of Connections (in Japanese), Kyoritsu Shuppan Co. (1957).

2) C. Ehresmann: Les connexions infinitésimales dans un espace fibré différentiable, Colloque de Topologie (Espaces fibrés), 29-55 (1950); S. S. Chern: Lecture note on differential geometry, Chicago University (1950).

3) See C. Ehresmann: Les prolongements d'une variété differentiable I. Calcul des jets, prolongement principal, C. R. Acad. Sci., Paris, **233**, 598-600 (1951).

$$\alpha=\sigma(\alpha)\eta(\alpha). \tag{1.5}$$

We get easily from the above formulas the following lemmas.

Lemma 1. *For any* $\alpha\in\mathfrak{L}_n^2$, $\beta\in\mathfrak{N}_n^2$, *we have*

$$a_{ik}^j(\alpha^{-1}\beta\alpha)=a_m^j(\alpha^{-1})\,a_{hl}^m(\beta)\,a_i^h(\alpha)\,a_k^l(\alpha). \tag{1.6}$$

Lemma 2. *For any* $\alpha,\alpha_1\in\mathfrak{L}_n^2$, *we have*

$$a_{ik}^j(\eta(\alpha))=a_h^j(\alpha^{-1})\,a_{ik}^h(\alpha), \tag{1.7}$$

$$a_{ik}^j(\eta(\alpha\alpha_1))=a_{ik}^j(\alpha_1^{-1}\eta(\alpha)\alpha_1)+a_{ik}^j(\eta(\alpha_1)). \tag{1.8}$$

2. The tangent space and associated principal bundle of order 2. For any differentiable manifold $\mathfrak{X}$ of dimension n, we shall define the tangent space $\mathfrak{T}^2(\mathfrak{X})$ of order 2 which will contain the tangent space $T(\mathfrak{X})$ in the ordinary sense. Let (u^i), $i=1,\cdots,n$, be a system of local coordinates of $\mathfrak{X}$ defined on an open neighborhood U. With the coordinate neighborhood (U,u^i), we associate $n+n^2$ fields of vectors X_i, X_{ik} defined on U. Let Y_i, Y_{ik} be the vector fields associated with another coordinate neighborhood (V,v^i). If $U\frown V\neq 0$, we assume that they are related mutually as

$$X_i=\frac{\partial v^j}{\partial u^i}\,Y_j, \tag{2.1}$$

$$X_{ik}=\frac{\partial^2 v^j}{\partial u^k\partial u^i}\,Y_j+\frac{\partial v^j}{\partial u^i}\,\frac{\partial v^h}{\partial u^k}\,Y_{jh}. \tag{2.2}$$

These formulas easily show that, at any point x of $\mathfrak{X}$, these vectors define a vector space of dimension $n+n^2$ independent of local coordinates. We call it the tangent space of order 2 of $\mathfrak{X}$ at the point x and denote it by $\mathfrak{T}_x^2(\mathfrak{X})$. This is, in fact, wider than the one $T_x^2(\mathfrak{X})$[4] of C. Ehresmann which may be obtained by putting $X_{ik}=X_{ki}$. The union

$$\mathfrak{T}^2(\mathfrak{X})=\bigcup_{x\in\mathfrak{X}}\mathfrak{T}_x^2(\mathfrak{X})$$

may be considered naturally as the total space of a vector bundle $\{\mathfrak{T}^2(\mathfrak{X}),\mathfrak{X},\bar{\tau}\}$ with the natural projection $\bar{\tau}$, whose structure group is $\mathfrak{L}_n^2$. For brevity, we denote also the vector bundle by the same notation $\mathfrak{T}^2(\mathfrak{X})$.

Let $\{\mathfrak{B}^2(\mathfrak{X}),\mathfrak{X},\bar{\pi}\}$ be the associated principal bundle of $\mathfrak{T}^2(\mathfrak{X})$. Any point $\bar{b}$ of $\mathfrak{B}^2(\mathfrak{X})$ may be regarded as a frame of $\mathfrak{T}^2(\mathfrak{X})$ at the point $\bar{\pi}(\bar{b})$ such that

$$\mathrm{e}_i(\bar{b})=X_j a_i^j(\bar{\alpha}),$$

$$\mathrm{e}_{ik}(\bar{b})=X_h a_{ik}^h(\bar{\alpha})+X_{jh}a_i^j(\bar{\alpha})a_k^h(\bar{\alpha}),$$

where $\bar{\alpha}\in\mathfrak{L}_n^2$. Corresponding to each $\bar{\alpha}\in\mathfrak{L}_n^2$, we define the right translation $r(\bar{\alpha})$ on $\mathfrak{B}^2(\mathfrak{X})$ by

$$\mathrm{e}_i(\bar{b}\bar{\alpha})=\mathrm{e}_j(\bar{b})a_i^j(\bar{\alpha}), \tag{2.3}$$

$$\mathrm{e}_{ik}(\bar{b}\bar{\alpha})=\mathrm{e}_j(\bar{b})a_{ik}^j(\bar{\alpha})+\mathrm{e}_{jh}(\bar{b})a_i^j(\bar{\alpha})a_k^h(\bar{\alpha}), \tag{2.4}$$

4) See the first reference in 2).

where we denote $r(\bar{\alpha})(\bar{b})$ simply by $\bar{b}\bar{\alpha}$.

By (2.1), we can define a natural imbedding $\iota: T(\mathfrak{X}) \to \mathfrak{T}^2(\mathfrak{X})$ by

$$\iota \frac{\partial}{\partial u^i} = X_i$$

and so we may identify X_i with $\partial/\partial u^i$. Let $\{\mathfrak{B}(\mathfrak{X}), \mathfrak{X}, \pi\}$ be the principal bundle of the tangent bundle $T(\mathfrak{X})$. Any point b of $\mathfrak{B}(\mathfrak{X})$ may be regarded as a frame of $T(\mathfrak{X})$ at the point $\pi(b)$ such that

$$\mathrm{e}_i(b) = a_i^j(\alpha)\frac{\partial}{\partial u^j}.^{5)}$$

Then, we can define a natural homomorphism $\sigma: \mathfrak{B}^2(\mathfrak{X}) \to \mathfrak{B}(\mathfrak{X})$ by

$$\iota(\mathrm{e}_i(\sigma(\bar{b}))) = \mathrm{e}_i(\bar{b}).$$

By this definition, it follows that $\bar{\pi} = \pi \cdot \sigma$ and $\tau = \bar{\tau} \cdot \iota$.

By virtue of (2.3), we see that $\mathfrak{B}(\mathfrak{X}) = \mathfrak{B}^2(\mathfrak{X})/\mathfrak{N}_n^2$ and $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$ is a vector bundle. Furthermore for any $\bar{\alpha} \in \mathfrak{L}_n^2$, we have easily

$$\sigma \cdot r(\bar{\alpha}) = r(\sigma(\bar{\alpha})) \cdot \sigma. \tag{2.5}$$

3. Connections. Theorem. *Any connection Γ of $T(\mathfrak{X})$ determines a cross section ρ of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$ invariant under the right translations. Conversely, such a cross section determines a connection Γ of $T(\mathfrak{X})$.*[6]

Proof. Let Γ_{ik}^j be the components of a given connection Γ of $T(\mathfrak{X})$ with respect to a coordinate neighborhood (U, u^i). For any $b \in \pi^{-1}(U)$, $\mathrm{e}_i(b) = a_i^j(\alpha)\dfrac{\partial}{\partial u^j}$, we put $\bar{b} = \rho(b)$ by

$$\mathrm{e}_i(\bar{b}) = \mathrm{e}_i(b), \quad \mathrm{e}_{ik}(\bar{b}) = (X_{jh} - \Gamma_{jh}^l X_l)a_i^j(\alpha)a_k^h(\alpha),$$

that is

$$\sigma(\bar{\alpha}) = \alpha, \quad a_{ik}^j(\bar{\alpha}) = -\Gamma_{hl}^j a_i^h(\alpha)a_k^l(\alpha), \tag{3.1}$$

from which we get easily the equations

$$a_{ik}^j(\eta(\bar{\alpha}^{-1})) = \Gamma_{ik}^j. \tag{3.2}$$

For another coordinate neighborhood (V, v^i), let β, $\bar{\beta}$ be the corresponding elements in L_n^1, $\mathfrak{L}_n^2$ respectively, then it must be

$$a_{ik}^j(\bar{\beta}) = -\left(\frac{\partial v^j}{\partial u^m}\Gamma_{ts}^m\frac{\partial u^t}{\partial v^h}\frac{\partial u^s}{\partial v^l} + \frac{\partial v^j}{\partial u^m}\frac{\partial^2 u^m}{\partial v^l \partial v^h}\right)a_i^h(\beta)a_k^l(\beta).$$

Since $a_i^j(\alpha) = \dfrac{\partial u^j}{\partial v^h}a_i^h(\beta)$, the equations above can be written as

$$\begin{aligned} a_{ik}^j(\bar{\beta}) &= -\frac{\partial v^j}{\partial u^m}\Gamma_{hl}^m a_i^h(\alpha)a_k^l(\alpha) + \frac{\partial^2 v^j}{\partial u^l u^h}a_i^h(\alpha)a_k^l(\alpha) \\ &= a_{ik}^j(g_{VU} \cdot \bar{\alpha}), \end{aligned}$$

where $g_{VU} \in \mathfrak{L}_n^2$ is the coordinate transformation of $\mathfrak{T}^2(\mathfrak{X})$ with respect to (U, u^i) and (V, v^i) such that

5) We will also use the same notation e_i in $T(\mathfrak{X})$ as in $\mathfrak{T}^2(\mathfrak{X})$, according to the above-mentioned consideration.

6) In the following, we shall consider only differentiable mappings with suitable differentiabilities.

$$a_i^j(g_{VU})=\frac{\partial v^j}{\partial u^i},\quad a_{ik}^j(g_{VU})=\frac{\partial^2 v^j}{\partial u^k \partial u^i}.$$

These equations show that ρ is well defined on the whole space $\mathfrak{B}(\mathfrak{X})$ as a cross section of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$.

Since we have $\mathrm{e}_i(b\alpha_0)=a^j(\alpha\alpha_0)\frac{\partial}{\partial u^j}$ for any $\alpha_0\in L_n^1$, we get easily

$$\mathrm{e}_i(\rho(b\alpha_0))=\mathrm{e}_i(b\alpha_0)=\mathrm{e}_j(b)a_i^j(\alpha_0)=\mathrm{e}_j(\rho(b))a_i^j(\alpha_0),$$
$$\mathrm{e}_{ik}(\rho(b\alpha_0))=(X_{jh}-\Gamma_{jh}^l X_l)a_i^j(\alpha\alpha_0)a_k^h(\alpha\alpha_0)$$
$$=\mathrm{e}_{jh}(\rho(b))a_i^j(\alpha_0)a_k^h(\alpha_0),$$

hence by (2.4) and $a_{ik}^j(\alpha_0)=0$ we obtain

$$\rho\cdot r(\alpha_0)=r(\alpha_0)\cdot\rho. \tag{3.3}$$

Conversely, let be given a cross section ρ of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$ satisfying (3.3). For the coordinate neighborhood (U, u^i), $b\in\pi^{-1}(U)$, $\bar{b}=\rho(b)$, we put

$$\mathrm{e}_{ik}(\bar{b})=X_j a_{ik}^j(\bar{\alpha})+X_{jh}a_i^j(\bar{\alpha})a_k^h(\bar{\alpha}).$$

By (3.3), we have $\rho(b\alpha_0)=\rho(b)\alpha_0$ for any $\alpha_0\in L_n^1$. Hence by means of (1.8), (1.6), we get

$$a_{ik}^j(\eta((\bar{\alpha}\alpha_0)^{-1}))=a_{ik}^j(\eta(\alpha_0^{-1}\bar{\alpha}^{-1}))$$
$$=a_{ik}^j(\bar{\alpha}\eta(\alpha_0^{-1})\bar{\alpha}^{-1})+a_{ik}^j(\eta(\bar{\alpha}^{-1}))=a_{ik}^j(\eta(\bar{\alpha}^{-1})).$$

This shows that

$$\Gamma_{ik}^j=a_{ik}^j(\eta(\bar{\alpha}^{-1}))$$

depends only on the coordinate neighborhood (U, u^i). We can easily prove that Γ_{ik}^j are the components of a connection Γ *of* $T(\mathfrak{X})$ with respect to the coordinate neighborhood. The proof is completed.

By virtue of this theorem, we may regard an affine connection of $\mathfrak{X}$ as an invariant cross section of $\{\mathfrak{B}^2(\mathfrak{X}), \mathfrak{B}(\mathfrak{X}), \sigma\}$. Let ρ, ρ_1 be the invariant cross sections corresponding to any two given connections Γ, Γ_1 of $T(\mathfrak{X})$ respectively. Then we define a mapping $\xi:\mathfrak{B}(\mathfrak{X})\to\mathfrak{N}_n^2$ by

$$\rho_1(b)=\rho(b)\xi(b). \tag{3.4}$$

By (3.3), (3.4), we get easily, for any $\alpha\in L_n^1$, $\xi(b\alpha)=\alpha^{-1}\xi(b)\alpha$ or

$$\xi\cdot r(\alpha)=A(\alpha^{-1})\cdot\xi.^{7)} \tag{3.5}$$

4. Connections of the type $(\mathfrak{Z}, \zeta)$. Let $\mathfrak{Z}$ be a subgroup (linear subspace) of $\mathfrak{N}_n^2$ and $Z=Z_{\mathfrak{Z}}$ be the subgroup of L_n^1 of all elements α such that $\alpha\mathfrak{Z}\alpha^{-1}=\mathfrak{Z}$. Let $\zeta:\mathfrak{X}\to\mathfrak{B}/Z$ be a cross section of the fibre bundle $\{\mathfrak{B}/Z, \mathfrak{X}\}$, where $\mathfrak{B}=\mathfrak{B}(\mathfrak{X})$.

For any point $b\in\mathfrak{B}$, if we have $b\alpha^{-1}/Z=\zeta(\pi(b))$, then we put $\mathfrak{Z}(b)=\alpha^{-1}\mathfrak{Z}\alpha$. This definition is clearly independent of the choice of such $\alpha\in L_n^1$. For any $\beta\in L_n^1$, we get easily

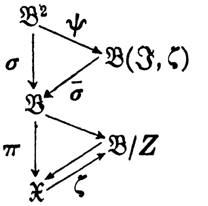
$$\mathfrak{Z}(b\beta)=\beta^{-1}\mathfrak{Z}(b)\beta. \tag{4.1}$$

Then, we can prove that the union

7) Here, we understand that $A(\alpha)$ denotes the inner automorphism of the group $\mathfrak{L}_n^2$ in the ordinary sense.

$$\bigcup_{b\in\mathfrak{B}}(\sigma^{-1}(b)/\mathfrak{J}(b))=\mathfrak{B}(\mathfrak{J},\zeta) \tag{4.2}$$

may be regarded naturally as a differentiable manifold. Let $\psi:\mathfrak{B}^2\to\mathfrak{B}(\mathfrak{J},\zeta)$ and $\bar{\sigma}:\mathfrak{B}(\mathfrak{J},\zeta)\to\mathfrak{B}$ be the natural projections. Thus, we obtain the diagram. $\{\mathfrak{B}^2, \mathfrak{B}(\mathfrak{J},\zeta), \psi\}$ and $\{\mathfrak{B}(\mathfrak{J},\zeta), \mathfrak{B}, \bar{\sigma}\}$ are fibre spaces.[8] Take any $\bar{b}\in\mathfrak{B}^2$, $\bar{\beta}\in\mathfrak{L}_n^2$ and put $b=\sigma(\bar{b})$, $\beta=\sigma(\bar{\beta})$. By means of (1.5), (4.1), we have $\bar{b}\mathfrak{J}(b)\bar{\beta}=\bar{b}\mathfrak{J}(b)\beta\eta(\bar{\beta})=\bar{b}\bar{\beta}\mathfrak{J}(b\beta)$. This equation shows that $r(\bar{\beta})$ may be considered as an operation defined on $\mathfrak{B}(\mathfrak{J},\zeta)$, which we shall denote by the same notation. Hence, we have



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$$\psi\cdot r(\bar{\alpha})=r(\bar{\alpha})\cdot\psi, \quad \bar{\alpha}\in\mathfrak{L}_n^2 \tag{4.3}$$

and

$$\bar{\sigma}\cdot r(\bar{\alpha})=r(\sigma(\bar{\alpha}))\cdot\bar{\sigma}. \tag{4.4}$$

Now, for any invariant cross section $\rho:\mathfrak{B}\to\mathfrak{B}^2$, the cross section

$$\bar{\rho}=\psi\cdot\rho \tag{4.5}$$

of the fibre space $\{\mathfrak{B}(\mathfrak{J},\zeta), \mathfrak{B}, \bar{\sigma}\}$ is also invariant under the group L_n^1 by means of (3.3) and (4.3), that is for any element $\alpha\in L_n^1$,

$$\bar{\rho}\cdot r(\alpha)=r(\alpha)\cdot\bar{\rho}. \tag{4.6}$$

We will say that any invariant cross section $\bar{\rho}$ of $\{\mathfrak{B}(\mathfrak{J},\zeta), \mathfrak{B}, \bar{\sigma}\}$ defines *a connection of the type* ($\mathfrak{J}$) *with respect to the cross section* $\zeta:\mathfrak{X}\to\mathfrak{B}/Z_{\mathfrak{J}}$ and simply call it a ($\mathfrak{J},\zeta$)-*connection of* $\mathfrak{X}$. If there exists an invariant cross section $\rho:\mathfrak{B}\to\mathfrak{B}^2$ such that $\bar{\rho}=\psi\cdot\rho$, we call it *an affine representative* of $\bar{\rho}$.

Now, when $\mathfrak{J}$ is invariant under any inner automorphism $A(\alpha)$, $\alpha\in L_n^1$, we have $Z_{\mathfrak{J}}=L_n^1$. Hence ζ is always the identity transformation on $\mathfrak{X}$ and $\mathfrak{B}(\mathfrak{J},\zeta)=\mathfrak{B}^2/\mathfrak{J}$.

Example 1. When $\mathfrak{J}=\{e\}$, a ($\mathfrak{J}$)-connection is a connection of $T(\mathfrak{X})$. When $\mathfrak{J}=\mathfrak{N}_n^2$, a ($\mathfrak{J}$)-connection is trivial, since $\mathfrak{B}(\mathfrak{J},\zeta)=\mathfrak{B}$.

Example 2. When $\mathfrak{J}=\{\beta\mid a_{ik}^j(\beta)=\delta_i^jp_k+p_i\delta_k^j\}$, a ($\mathfrak{J}$)-connection is clearly a projective connection in the ordinary sense.

Lastly, we shall give two examples such that $\mathfrak{J}$ is not invariant.

Example 3. When $\mathfrak{J}=\{\beta\mid a_{ik}^j(\beta)=\delta_i^jp_k+p_i\delta_k^j-\delta_{ik}p_j\}$, $Z=Z_{\mathfrak{J}}$ is the subgroup of L_n^1 under which the equation $\sum x^ix^i=0$ is invariant, that is, the Euclidean angle is invariant. $\mathfrak{B}/Z$ is the space of all conical surfaces of signature n in the tangent space at each point of $\mathfrak{X}$. Hence a cross section $\zeta:\mathfrak{X}\to\mathfrak{B}/Z$ is a field of such conical surfaces over $\mathfrak{X}$. Accordingly, a ($\mathfrak{J},\zeta$)-connection is a sort of conformal connections and the conformal connections in the ordinary sense are the one determined from ζ by a rule such that the angles measured by ζ are

8) $\{\mathfrak{B}^2, \mathfrak{B}(\mathfrak{J},\zeta), \psi\}$ is not a principal fibre bundle in the ordinary sense but it becomes so when $\mathfrak{J}$ is invariant under the group L_n^1. See N. Steenrod: The Topology of Fibre Bundle, Princeton, §8 (1951).

always invariant under parallel displacement along any curve with respect to any one of its affine representatives.

Example 4. When $\mathfrak{J}=\{\beta \mid a^j_{ik}(\beta)=\delta^j_i\delta^n_k p\}$, $Z=Z_{\mathfrak{J}}$ is the subgroup of L^1_n under which the coordinate hyperplane $x^n=0$ in R^n is invariant. $\mathfrak{B}/Z$ is the space of all cotangent directions of $\mathfrak{X}$. Hence a cross section $\zeta:\mathfrak{X}\rightarrow\mathfrak{B}/Z$ is a field of $(n-1)$-dimensional tangent subspaces of $\mathfrak{X}$. In this case, we can prove the following proposition.

Proposition. *In order that two affine connections are representatives of a $(\mathfrak{J},\zeta)$-connection, it is necessary and sufficient that*

(i) *any field of tangent directions of $\mathfrak{X}$ defined on any curve has the same development with respect to the two connections and*

(ii) *the induced connections from the two connections on any curve tangent to ζ at each of its points coincide with each other.*