92. An Investigation on the Logical Structure of Mathematics. XII[1]

The Principle of Extensionality and of Choice

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1. Principle of extensionality. The principle of extensionality means, roughly speaking, that a "set" is determined by its elements, or that two "sets" which have their elements in common are equal. On the other hand, if two "sets" are equal, they must have their properties in common. Accordingly, $a=b$ can be looked upon as the abbreviation of either one of the following formulas:

(1) $$\forall x.x\in a\equiv x\in b,$$

(2) $$\forall x.a\in x\equiv b\in x.$$

The former was adopted in UL$^{(I)}$, and therefore the formula

(I) $$\forall xyz.x=y\wedge x\in z\Rightarrow y\in z$$

expresses in UL the principle of extensionality. On the contrary, if we adopt the latter, the principle of extensionality is expressed by

(J) $$\forall xyz.x=y\wedge z\in x\Rightarrow z\in y.$$

Let, now, P^c be the dependent variable in UL defined by

(3) $$\forall u.u\in P^c\equiv c\in u.$$

Namely, P^c is the "set" of all properties of c. By using P^c as a set$^{(X)}$ in a UL-proof, we shall prove that "(2) implies (1)" is a UL-theorem with (3) as unique premise. To prove this, we have only to give a UL-proof with (3) as premise and with

(4) $$\forall x.a\in x\equiv b\in x.\wedge c\in a:\Rightarrow c\in b$$

as conclusion, since it is clear that no contradiction can be deduced from (3).[2] The UL-proof is simply as follows:

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1) For the present state of publication of this investigation, see footnote 0) in Part (IV), Nagoya Math. J., **13** (1958). The subtitles of Parts (I)-(XI) are as follows: (I) A logical system; (II) Transformation of proof; (III) Fundamental deductions; (IV) Compendium for deductions; (V) Contradictions of Russell's type; (VI) Consistent V-system; (VII) Set-theoretical contradictions; (VIII) Consistency of the natural number theory $T_1(N)$; (IX) Deduction of natural number theory in $T_1(N)$; (X) Concepts and sets; (XI) Underlying ideas of the investigation (in Japanese), The indices (I), (II), etc. attached in the following are the references to other Parts.

The contents of Parts (I)-(XI) were verbally published at the spring and autumn meetings, 1957, of the Mathematical Society of Japan and at the spring meeting, 1958, of the Japan Association for Philosophy of Science.

2) For P^c is a T(0)- as well as T(V)-set$^{(VI)}$. We can use, instead of P^c, the variable Q^c defined by $\forall u.u\in Q^c\equiv c\notin u$ to prove (4).

$$
\begin{array}{l}
- \qquad (4) \\
\hline
- \quad \neg \forall x . a \in x \equiv b \in x \\
1 \quad c \notin a \\
2 \quad c \in b \\
\hdashline
- \quad a \in \mathrm{P}^c \not\equiv b \in \mathrm{P}^c \\
\hline
\begin{array}{ll}
- \; a \in \mathrm{P}^c & - \; b \notin \mathrm{P}^c \\
\hline
c \in a & c \notin b \\
(1) & (2)
\end{array}
\end{array}
$$

The formula (4) means that if the equality is looked upon as the abbreviation of (2), the principle of extensionality (J) is a UL-theorem. Namely, the principle of extensionality is replaced by the defining formula (3) of P^c. To adopt this definition of equality in defining UL is, however, not practical for the actual deduction of mathematics, since the formula (1) is required for equality in most cases in the proofs. However, a UL-proof can easily be changed into a UL-proof in which equality is the abbreviation of (2). Namely, wherever $a=b$ is changed into (1) in a UL-proof, we replace (1) by (2) and insert directly under it an ordinary cut[(IV)] of which $(2) \Rightarrow (1)$ is one of the cut formulas, adjoining moreover (3) as a premise, and wherever a proof constituent associated to (I) is used we replace the proof constituent by a cut with the cut formula (4), erasing the premise (I) from the proof. Thus, the original UL-proof is transformed to a UL-proof in which equality $a=b$ is the abbreviation of (2) and the premises consist exclusively of defining formulas.

2. Principle of choice. We define a dependent variable Ch^a (choice) by[3)]

$$
(5) \qquad \forall u . u \in \mathrm{Ch}^a \equiv u \in \mathrm{Un} \wedge \overset{x \neq 0}{\underset{a}{\forall}} x \overset{x}{\exists} y . \langle xy \rangle \in u .
$$

Namely, Ch^a is the "set" of all choice functions of a, of which the domain of definition contains all non-void elements of a. The universal choice Ch^{V}, denoted simply by Ch, is defined by the specialized defining formula[§20,(II)] of (5), namely by

$$
(6) \qquad \forall u . u \in \mathrm{Ch} \equiv u \in \mathrm{Un} \wedge \overset{x \neq 0}{\forall} x \overset{x}{\exists} y . \langle xy \rangle \in u .
$$

Now, the principle of choice for a or for V is expressed, respectively, by $\mathrm{Ch}^a \neq 0$ or $\mathrm{Ch} \neq 0$. The expression $\mathrm{Ch}^a \neq 0$, and $\mathrm{Ch} \neq 0$, as well as $x \neq 0$ in (5) and (6), are expressed by the characteristic property[§20,(II)] of 0, namely by $\exists x . x \in \mathrm{Ch}^a$, $\exists x . x \in \mathrm{Ch}$, and $\exists y . y \in x$, respectively, so that in using the principle of choice the null constant 0 is only used as concept[(X)]. Hence, the dependent variables Ch^a and Ch are T(V)-sets[(VI)],

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3) $\overset{x \neq 0}{\underset{a}{\forall}} x$ and $\overset{x}{\exists} y$ denote the restriction of quantifiers, namely, $\forall x \; x \in a \; x \neq 0 \Rightarrow$ and $\exists y . x \in y \wedge$, respectively.

although there is until now certainly no branch of mathematics in which Ch^a or Ch is used as set[(X)].

In dealing with a UL-assertion $\sigma \vdash H$ or the proof of $\sigma \vdash H$ in which the principle of choice is required, we place the defining formula (5) or (6) in σ and, on the contrary, the formula $\mathrm{Ch}^a \neq 0$ or $\mathrm{Ch} \neq 0$ as an assumption[4] of the conclusion H. Thus, if the principle of choice is allowed in a UL-theory T, then the consistency of T under the assumption $\mathrm{Ch}^a \neq 0$ or $\mathrm{Ch} \neq 0$ amounts to the T-unprovability of $\mathrm{Ch}^a = 0$ or $\mathrm{Ch} = 0$, just in the same way as the $\mathrm{T}_1(\tilde{\mathrm{N}})$-unprovability of $\tilde{\mathrm{N}} = \mathrm{V}$[(VI)].

In this way, both principles of extensionality and choice are replaced by defining formulas, so that the premises of UL-assertion can be considered to consist exclusively of defining formulas. By this improvement concerning the foundations of the universal logic UL, it acquires much more universal character than has been considered since in my previous papers.[5]

3. Structure of mathematics. I define *logic* as well as *mathematics* as UL. So I share with Russell, though not in the same sense as his, the thesis that logic and mathematics are identical.[6] The structure of mathematics is, therefore, the structure of UL.

The concepts, propositions, and inferences are the three main subjects of logic. Accordingly, the variables, formulas, and proofs of UL are the three main subjects of UL.

Among others, the dependent variables are defined by their defining formulas. However, some properties of a dependent variable are not intrinsic to itself but depending on what other dependent variables co-exist with it. For instance, a dependent variable, co-existing with some other dependent variables, has sometimes a finite number of elements and sometimes infinite, or sometimes constitutes consistent subsystems of UL and sometimes gives rise to contradictions[(VII)]. The investigation of the nature of such contradictions and that of such consistent systems are equally important problems con-

4) When a conclusion H of a UL-assertion $\sigma \vdash H$ has the form $A_1 \wedge \cdots \wedge A_n \Rightarrow B$, we call A_i $(1 \leq i \leq n)$ an *assumption* of the conclusion H, distinguishing it from the premises σ of the assertion $\sigma \vdash H$.

5) At first, the author's intention was to formulate the principle of choice by some special formula placed in the premises, just in the same way as in the case of the principle of extensionality. By the above description both principles are replaced by the defining formulas, so that the several comments, scattered in Parts (I), (II), and (VII) concerning the axiom of choice, turn out unnecessary, and the potential necessity of amending the foundations of UL, which might perhaps arise by adjoining to premises a formula expressing the principle of choice, has also disappeared. In particular, the system ULM, provisionally mentioned in the introduction in Part (II) is proved to be a subsystem of UL, not a system lying outside UL.

6) See, for instance, B. Russell: Principles of Mathematics, 2nd ed., Introd., 5 (1937).

cerning the structure of UL, i.e. of mathematics or of logic.

By the description in §1 and §2 the UL-premises can be put into the unique scheme

$$(7) \qquad \forall x_1 \cdots x_n \exists p \forall u . u \in p \equiv F^{u; x_1, \cdots, x_n},$$

where F is any well-formed formula construed with the unique dyadic relation $\in$ and with $u, x_1, \cdots, x_n$ as the complete system of free variables in F. In using (7), however, a particular attention is needed to the above-mentioned peculiar properties of the dependent variables, and moreover, there is some disadvantage in formulating the conclusion of a UL-assertion. This is a reason why UL is defined by the recursive method of introducing dependent variables into UL, which is also an everyday way in mathematics. For instance, Bourbaki's Éléments de Mathématique are formulated in this way. Moreover, Bourbaki's "structure de l'espèce T"[7] corresponds to a subsystem of UL.

7) See N. Bourbaki: Théorie des Ensembles (Fascicule de résultats), 2me éd., §8 (1951).