139. On the Cohomology Groups of $\mathfrak{p}$-adic Number Fields

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In the present note we shall study the cohomology groups of the ring of all $\mathfrak{p}$-integers of a $\mathfrak{p}$-adic field.

Let K be a $\mathfrak{p}$-adic number field and let L be a finite separable extension field over K. More generally, let K be a complete field by a discrete valuation and let L be a finite separable extension field over K with separable residue class field. Let R and Λ be the rings of all $\mathfrak{p}$-integers of K and L, respectively. Then Λ has a minimal basis over R, i.e.

$$\Lambda=R+R\theta+\cdots+R\theta^{n-1},$$

where $1, \theta, \cdots, \theta^{n-1}$ are linearly independent over R [1]. Let $f(x)=0$ be the equation of θ in R.

We shall consider Λ as an algebra over R and construct a Λ^e-projective resolution over Λ which is suitable for our purpose.

Let

$$f(x)=(x-\theta)g(x), \quad g(x)=x^{n-1}+(\sum_j b_{n-2,\,j}\theta^j)x^{n-2}+\cdots$$

be the decomposition of $f(x)$ in Λ. We put

$$g_e(\theta)=\sum_{i,j} b_{ij}\,\theta^i\otimes\theta^j \text{ [1]}$$

$$\Delta\theta=\theta\otimes1-1\otimes\theta$$

in $\Lambda^e=\Lambda\otimes_R\Lambda$.

Lemma

Let $\sum\lambda\otimes\mu$ *be any element in* Λ^e. *Then*

$(\sum\lambda\otimes\mu)(\theta\otimes1-1\otimes\theta)=0$ *if and only if* $\sum\lambda\otimes\mu\in\Lambda^e\cdot g_e(\theta)$;

$(\sum\lambda\otimes\mu)\cdot g_e(\theta)=0$ *if and only if* $\sum\lambda\otimes\mu\in\Lambda^e(\theta\otimes1-1\otimes\theta)$.

Proof. Since we have a ring isomorphism

$$\Lambda\otimes_R\Lambda\cong\Lambda[x]/(f(x)),$$

$$\theta\otimes1-1\otimes\theta \longleftrightarrow x-\theta \mod (f(x)),$$

$$g_e(\theta) \longleftrightarrow g(x) \mod (f(x)),$$

we shall calculate in the right hand side. We take polynomials of degree less than n as the uniquely determined representatives of the classes mod $f(x)$. If $(x-\theta)h(x)\equiv0$ mod $f(x)$, deg $h(x)\leqq n-1$, then dividing $h(x)$ by $g(x)$ we have $h(x)=\alpha g(x)+s(x)$, deg $s(x)\leqq n-2$; so $s(x)(x-\theta)\equiv0$ mod $f(x)$. Therefore $s(x)=0$, $h(x)=\alpha g(x)$. Similarly, if $g(x)h(x)\equiv0$ mod $f(x)$, then $h(x)=(x-\theta)h_0(x)$.

Lemma

1) Since Λ is commutative, $\Lambda^*\cong\Lambda$ and we shall drop the sign $*$.

The kernel of the augmentation $\varepsilon: \Lambda^e \to \Lambda$, $\varepsilon(\lambda \otimes \mu) = \lambda\mu$ *is* $\Lambda^e(\theta \otimes 1 - 1 \otimes \theta)$.

Proof. Since Λ is commutative, ε is a ring homomorphism. So that $\Lambda^e(\theta \otimes 1 - 1 \otimes \theta)$ is contained in the kernel of ε. Conversely, if $\varepsilon(\sum_{i,j} c_{i,j}\theta^i \otimes \theta^j) = 0$, then from $\sum_{i,j} c_{i,j}\theta^i \otimes \theta^j$

$$=\sum c_{ij}(\theta^i \otimes 1)(1 \otimes \theta^j) = \sum c_{ij}(\theta^i \otimes 1)\{\theta^j \otimes 1 + (1 \otimes \theta^j - \theta^j \otimes 1)\}$$
$$=\sum c_{ij}(\theta^i \otimes 1)\{\theta^j \otimes 1 + (1 \otimes \theta^{j-1} + \theta \otimes \theta^{j-2} + \cdots + \theta^{j-1} \otimes 1)(1 \otimes \theta - \theta \otimes 1)\}$$
$$=\sum c_{ij}\theta^{i+j} \otimes 1 + [\sum c_{ij}(\theta^i \otimes 1)(1 \otimes \theta^{j-1} + \cdots + \theta^{j-1} \otimes 1)](1 \otimes \theta - \theta \otimes 1)$$

we have $\varepsilon((\sum c_{ij}\theta^{i+j}) \otimes 1) = \sum c_{ij}\theta^{i+j} = 0$, which proves the assertion.

Now we consider the following Λ^e-resolution over Λ:

$$\cdots \xrightarrow{d_4} \Lambda^e \xrightarrow{d_3} \Lambda^e \xrightarrow{d_2} \Lambda^e \xrightarrow{d_1} \Lambda^e \xrightarrow{\varepsilon} \Lambda \longrightarrow 0$$

where
$$\varepsilon: \Lambda^e \to \Lambda, \quad \varepsilon(\sum \lambda \otimes \mu) = \sum \lambda\mu$$
$$d_{2r+1}(\sum \mu \otimes \lambda) = (\sum \mu \otimes \lambda)(\theta \otimes 1 - 1 \otimes \theta)$$
$$d_{2r}(\sum \mu \otimes \lambda) = (\sum \mu \otimes \lambda) g_e(\theta).$$

This is Λ^e-free and, by the above lemma, acyclic.

To calculate $H^n(\Lambda, A)$ and $H_n(\Lambda, A)$ for any Λ^e module A, we consider the complex

$$\cdots \xleftarrow{\delta_3} \mathrm{Hom}_{\Lambda^e}(\Lambda^e, A) \xleftarrow{\delta_2} \mathrm{Hom}_{\Lambda^e}(\Lambda^e, A) \xleftarrow{\delta_1} \mathrm{Hom}_{\Lambda^e}(\Lambda^e, A)$$
$$\cdots \xrightarrow{\partial_3} A \otimes_{\Lambda^e} \Lambda^e \xrightarrow{\partial_2} A \otimes_{\Lambda^e} \Lambda^e \xrightarrow{\partial_1} A \otimes_{\Lambda^e} \Lambda^e$$

where δ_i and ∂_i are induced homomorphisms of d_i. Considering the isomorphisms

$$\mathrm{Hom}_{\Lambda^e}(\Lambda^e, A) \cong A, \quad A \otimes_{\Lambda^e} \Lambda^e \cong A,$$

we may translate δ_i and ∂_i into the endomorphisms of A

$$\partial_{2r+1}(a) = a\theta - \theta a (= \Delta^*\theta \cdot a), \quad \delta_{2r+1}(a) = \theta a - a\theta,$$
$$\partial_{2r}(a) = \sum b_{ij}\theta^j a \theta^i (= g_e^*(\theta) \cdot a), \quad \delta_{2r}(a) = \sum b_{ij}\theta^i a \theta^j$$

for $a \in A$. Thus we have

Theorem

$$H^{2r+1}(\Lambda, A) \cong A_{g_e(\theta)}/A^{\Delta\theta}, \quad H^{2r+2}(\Lambda, A) \cong A_{\Delta\theta}/A^{g_e(\theta)},$$
$$H_{2r+2}(\Lambda, A) \cong A_{g_e^*(\theta)}/A^{\Delta^*\theta}, \quad H_{2r+1}(\Lambda, A) \cong A_{\Delta^*\theta}/A^{g_e^*(\theta)}$$

for $r \geqq 0$, *where*

$$A_\square = \{a \in A \mid \square a = 0\}, \quad A^\square = \{\square a \mid a \in A\}$$

for any two sided Λ *module* A *(considered as left* Λ^e *module).*

Corollary

$$H^{n+2}(\Lambda, A) \cong H^n(\Lambda, A)$$
$$H_{n+2}(\Lambda, A) \cong H_n(\Lambda, A)$$

for $n \geqq 1$.

Theorem

If $\theta a = a\theta$ *for any* a *in* A, *then*

$$H^{2r+1}(\Lambda, A) \cong H_{2r+2}(\Lambda, A) \cong A_{f'(\theta)}$$
$$H^{2r+2}(\Lambda, A) \cong H_{2r+1}(\Lambda, A) \cong A/A^{f'(\theta)}. \quad r \geqq 0$$

Proof. In this case $g_e(\theta)\cdot a = g(\theta)a$ and

$$g(\theta)=(\theta-\theta')\cdots(\theta-\theta^{(n-1)})=f'(\theta).$$

The corollary of this note may be extended to the global case. Let K and L be the algebraic number fields, R and Λ the rings of all integers of K and L respectively. Then for any Λ^e-finitely generated module A we have

$$H_{n+2}(\Lambda, A)\cong H_n(\Lambda, A)$$
$$H^{n+2}(\Lambda, A)\cong H^n(\Lambda, A)$$

for $n\geqq 1$. We may prove it by reducing it to the $\mathfrak{p}$-component and by using the above corollary.

References

[1] E. Artin: Algebraic Numbers and Algebraic Functions I (mimeographed note), New York University (1951).

[2] H. Cartan and S. Eilenberg: Homological Algebra, Princeton (1956).