136. On the Singular Integrals. III*)

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1. Let $f(x)$ be a real or complex valued measurable function over $(-\infty, \infty)$, and $\tilde{f}(x)$ the Hilbert transform of $f(x)$, that is

$$\tilde{f}(x)=\lim_{\eta\to 0}\int_{|x-t|>\eta}\frac{f(t)}{x-t}dt. \tag{1.1}$$

The property of this singular integral has been studied by many authors. In particular we feel interest in the result of L. H. Loomis [5] and H. Kober [3]. Other references will be found in E. C. Titchmarsh [6].

We introduce instead of the ordinary Lebesgue measure the following one

$$\mu(\alpha, x)=\int_0^x\frac{dt}{1+|t|^\alpha}, \quad (\alpha\geqq 0) \tag{1.2}$$

and we consider the Hilbert transform for the function of the class $L^p_\mu(p\geqq 1)$ which is the set of $f(x)$ to be measurable with respect to the $d\mu$ and such as

$$\int_{-\infty}^{\infty}|f(x)|^p d\mu(x)=\int_{-\infty}^{\infty}\frac{|f(x)|^p}{1+|x|^\alpha}dx<\infty. \tag{1.3}$$

Clearly the measure function (1.2) plays a role of convergence factor and this enables us to treat the more extensive class of functions. The purpose of this paper is the systematic treatment of the Hilbert operator from a point of view of linear operation. We may use the same notation of constant at each occurrence.

2. The theorems on interpolation of the operation which we need are three. Definitions which we do not state here will be found in the paper of A. Zygmund [7]. Let R and S be two spaces—for simplicity Euclidean spaces—with non-negative and completely additive measure μ and ν respectively. In a previous paper the author [4, I] has extended the definition of A. Zygmund to the case where $\mu(R)$ and $\nu(S)$ are infinite. We need two more theorems which are due to A. P. Calderón and A. Zygmund [1]. These concern with the ordinary Lebesgue measure and we can state in the following form:

Theorem. *Suppose that $\mu(R)$ and $\nu(S)$ are both infinite and that a quasi-linear operation $\tilde{f}=Tf$ is simultaneously of weak type* (1,1) *and* (p, p) $(p>1)$. *Then $\tilde{f}=Tf$ is defined for every f such that*

*) Here we state the result without proof. The detailed argument will appear in the Jour. Faculty Sci. Hokkaidô Univ.

$|f|\log^+|f|\in L_\mu$ *and we have for any sub-set* S_0 *with finite* ν*-measure*

$$\int_{S_0}|\tilde{f}|\,d\nu \leqq A\int_R |f|\{1+\log^+[\nu(S_0)^\gamma|f|]\}d\mu + B\nu(S_0)^{1-\gamma}, \tag{2.1}$$

where γ *is any positive number.*

Theorem. *Suppose that* $\mu(R)$ *and* $\nu(S)$ *are both infinite, and that a quasi-linear operation* $\tilde{f}=Tf$ *is of weak type* $(1,1)$. *Then we have for any sub-set* S_0 *of* S *with finite* ν*-measure*

$$\int_{S_0}|\tilde{f}|^{1-\varepsilon}d\nu \leqq \frac{A}{\varepsilon}\nu(S_0)^\varepsilon\Big(\int_R |f|\,d\mu\Big)^{1-\varepsilon}, \tag{2.2}$$

where $0<\varepsilon<1$ *and* A *is an absolute constant.*

3. In this section we use a weak type or strong type of the operation in mean with respect to the measure $d\mu$. From now on we limit p and α such as $p>1$, $0\leqq\alpha<1$. The equivalence of the measurability in the ordinary Lebesgue sense and that of the $d\mu$ is clear.

Theorem 1. *Let* $f(x)$ *belong to* L_μ^p, *then the Hilbert operation* (1.1) *is a strong type* (p,p).

This theorem has been proved by the same argument of G. H. Hardy and J. E. Littlewood [2].

Theorem 2. *Let* $f(x)$ *belong to* L_μ. *Then the Hilbert operation* (1.1) *is a weak type* $(1,1)$.

This theorem has been proved by running on the line of A. Zygmund [loc. cit.], but we need the following

Lemma 1. *For any interval* $I=(a,b)$ *and* $0\leqq\alpha<1$, *there exist two contiguous interval* $I_*=(a',a)$ *and* $I^*=(b,b')$ *such that*

$$\mu(I_*)\leqq\mu(I^{(1)}),\quad \mu(I^*)\leqq\mu(I^{(2)}) \tag{3.1}$$

$$A_\alpha|I|/2\leqq|I_*|\leqq A'_\alpha|I|/2 \tag{3.2}$$

$$B'_\alpha|I|/2\leqq|I^*|\leqq B_\alpha|I|/2 \tag{3.3}$$

where

$$I^{(1)}=(a,c),\quad I^{(2)}=(c,b),\quad 2c=a+b, \tag{3.4}$$

and by $|I|$ *we denote the length of the interval* I.

The case $\alpha=0$ in Theorems 1 and 2 is due to M. Riesz [cf. 6] and L. H. Loomis [5] respectively. In these theorems the operation can be replaced by the following

$$Tf=\tilde{f}_\eta(x)=\frac{1}{\pi}\int_{|x-t|>\eta}\frac{f(t)}{x-t}dt, \tag{3.5}$$

where $\eta=\eta(x)$ is any positive measurable function of x. In particular we can put $\sup_{\eta>0}|\tilde{f}_\eta(x)|$.

4. If we apply interpolating theorems to the result of the preceding section we have immediately

Theorem 3. *Let* $f(x)$ *belong to* L_μ^p $(0\leqq\alpha<1)$. *Then the Hilbert operation* (1.1) *exists for a.e. and also belong to the same class and we* *have*

$$\int_{-\infty}^{\infty}\frac{\varphi(|\tilde{f}|)}{1+|x|^{\alpha}}dx \leqq A\int_{-\infty}^{\infty}\frac{\varphi(|f|)}{1+|x|^{\alpha}}dx \tag{4.1}$$

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{\varphi(|\tilde{f}-\tilde{f}_{\eta}|)}{1+|x|^{\alpha}}dx=0. \tag{4.2}$$

In particular we have

Corollary 1. *Let $f(x)$ belong to L_{μ}^{p} $(p>1,\ 0\leqq\alpha<1)$ then we have*

$$\int_{-\infty}^{\infty}\frac{|\tilde{f}|^{p}}{1+|x|^{\alpha}}dx \leqq A_{p,\alpha}\int_{-\infty}^{\infty}\frac{|f|^{p}}{1+|x|^{\alpha}}dx \tag{4.3}$$

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{f}-\tilde{f}_{\eta}|^{p}}{1+|x|^{\alpha}}dx=0. \tag{4.4}$$

Theorem 4. *Let $f(x)$ be a function such that*

$$\int_{-\infty}^{\infty}\frac{|f|\log^{+}[(1+x^{2})|f|]}{1+|x|^{\alpha}}dx<\infty, \quad (0<\alpha<1). \tag{4.5}$$

Then the Hilbert operation is integrable on the whole interval and we have

$$\int_{-\infty}^{\infty}\frac{|\tilde{f}|}{1+|x|^{\alpha}}dx \leqq A\int_{-\infty}^{\infty}\frac{|f|\log^{+}[(1+x^{2})|f|]}{1+|x|^{\alpha}}dx+B \tag{4.6}$$

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{f}-\tilde{f}_{\eta}|}{1+|x|^{\alpha}}dx=0, \tag{4.7}$$

where A, B are absolute constants.

Theorem 5. *Let $f(x)$ be a function such that*

$$\int_{-\infty}^{\infty}|f|\log^{+}[(1+x^{2})|f|]dx<\infty. \tag{4.8}$$

Then we have

$$\int_{-\infty}^{\infty}|\tilde{F}|dx \leqq A\int_{-\infty}^{\infty}|f|\log^{+}[(1+x^{2})|f|]dx+B \tag{4.9}$$

$$\lim_{\lambda,\eta\to 0}\int_{-\infty}^{\infty}|\tilde{F}_{\lambda}-\tilde{F}_{\eta}|dx=0 \tag{4.10}$$

where

$$\tilde{F}_{\eta}(x)=\tilde{f}_{\eta}(x)-\frac{\mathrm{K}_{1}(x)}{\pi}\int_{-\infty}^{\infty}f(t)dt, \tag{4.11}$$

where A, B are absolute constants.

Theorem 6. *Let $f(x)$ belong to L_{μ} $(0\leqq\alpha<1)$. Then we have*

$$\int_{-\infty}^{\infty}\frac{|\tilde{f}|^{1-\varepsilon}}{1+|x|^{\alpha+\delta}}dx \leqq \frac{A}{\varepsilon\{\delta-\varepsilon(1-\alpha)\}}\left(\int_{-\infty}^{\infty}\frac{|f|}{1+|x|^{\alpha}}dx\right)^{1-\varepsilon} \tag{4.12}$$

$$\lim_{\eta\to 0}\int_{-\infty}^{\infty}\frac{|\tilde{f}-\tilde{f}_{\eta}|^{1-\varepsilon}}{1+|x|^{\alpha+\delta}}dx=0 \tag{4.13}$$

where $0<\varepsilon<1$, $\delta>\varepsilon(1-\alpha)$ *and* A *is an absolute constant.*

5. For any sequence

$$X=(\cdots, x_{-1}, x_0, x_1, \cdots, x_n, \cdots) \tag{5.1}$$

we define the Hilbert transform by the following sequence

$$\tilde{X}=(\cdots, \tilde{x}_{-1}, \tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_n, \cdots) \tag{5.2}$$

where

$$\tilde{x}_n = \sum_{m=-\infty}^{\infty}{}' \frac{x_m}{n-m} \tag{5.3}$$

the prime means that the term $m=n$ is omitted from summation.

If we put

$$x(u)=x_n \text{ if } |n-u|\leqq 1/2 \ (n=0, \pm 1, \cdots) \tag{5.4}$$

and similarly for $\tilde{x}_n$. Then a transformation $\tilde{x}(u)=Tx(u)$ defines a linear operation. The weak type $(1,1)$ and strong type (p,p) of this operation are equivalent to the following propositions

$$\sum_{n:|\tilde{x}_n|>y} \frac{1}{1+|n|^\alpha} \leqq \frac{M}{y} \sum_{-\infty}^{\infty} \frac{|x_n|}{1+|n|^\alpha} \tag{5.5}$$

and

$$\sum_{-\infty}^{\infty} \frac{|\tilde{x}_n|^p}{1+|n|^\alpha} \leqq A_{p,\alpha} \sum_{-\infty}^{\infty} \frac{|x_n|^p}{1+|n|^\alpha}, \tag{5.6}$$

respectively. Thus we can reduce the discrete case to the continuous case of the preceding section. We have

Theorem 7. *Let* X *belong to* l^φ_μ. *Then* $\tilde{X}$ *also belongs to the same class and we have*

$$\|\tilde{X}\|_{\varphi,\mu} \leqq A_{\varphi,\alpha} \|X\|_{\varphi,\mu}. \tag{5.7}$$

Corollary 2. *Let* X *belong to* l^p_μ $(p>1,\ 0\leqq\alpha<1)$. *Then we have*

$$\|\tilde{X}\|_{p,\mu} \leqq A_{p,\alpha} \|X\|_{p,\mu}. \tag{5.8}$$

Theorem 8. *Let* X *be a sequence such that*

$$\sum_{-\infty}^{\infty} \frac{|x_n| \log^+[(1+n^2)|x_n|]}{1+|n|^\alpha} < \infty \tag{5.9}$$

where $0<\alpha<1$. *Then* $\tilde{X}$ *can be defined and we have*

$$\sum_{-\infty}^{\infty} \frac{|\tilde{x}_n|}{1+|n|^\alpha} \leqq A \sum_{-\infty}^{\infty} \frac{|x_n| \log^+[(1+n^2)|x_n|]}{1+|n|^\alpha} + B \tag{5.10}$$

where A, B *are absolute constants.*

Theorem 9. *Let* X *be a sequence such that*

$$\sum_{-\infty}^{\infty} |x_n| \log^+[(1+n^2)|x_n|] < \infty. \tag{5.11}$$

Then we have

$$\sum_{-\infty}^{\infty} |\tilde{x}^*_n| \leqq A \sum_{-\infty}^{\infty} |x_n| \log^+[(1+n^2)|x_n|] + B \tag{5.12}$$

where

$$\tilde{x}_0^* = \tilde{x}_0 \quad and \quad \tilde{x}_n^* = \tilde{x}_n - \frac{1}{n} \sum_{-\infty}^{\infty} x_n, \ (n = \pm 1, \pm 2, \cdots). \tag{5.13}$$

Theorem 10. *Let X belong to the class L_μ $(0 \leqq \alpha < 1)$. Then $\tilde{X}$ can be defined and we have*

$$\sum_{-\infty}^{\infty} \frac{|\tilde{x}_n|^{1-\varepsilon}}{1+|n|^{\alpha+\delta}} \leqq \frac{A}{\varepsilon\{\delta - \varepsilon(1-\alpha)\}} \Big(\sum_{-\infty}^{\infty} \frac{|x_n|}{1+|n|^\alpha} \Big)^{1-\varepsilon} \tag{5.14}$$

where $0 < \varepsilon < 1$, $\delta > \varepsilon(1-\alpha)$ and A is an absolute constant.

6. As a simple application we establish some theorems concerning the Dirichlet singular integral. This is defined by the following formula:

$$D_\lambda(x, f) = D_\lambda(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \lambda(x-t)}{x-t} dt. \tag{6.1}$$

This can be rewritten in the following form,

$$D_\lambda(x) = \frac{\sin \lambda x}{\pi} \int_{-\infty}^{\infty} \frac{f(t) \cos \lambda t}{x-t} dt - \frac{\cos \lambda x}{\pi} \int_{-\infty}^{\infty} \frac{f(t) \sin \lambda t}{x-t} dt. \tag{6.2}$$

Thus we have similar theorems. We omit the detail.

7. In the final section we give a negative example:

Theorem 11. *For any given pair of numbers $(p > 1, \alpha \geqq 1)$ or $(p = 1, \alpha > 1)$, there is a function of class L_μ^p whose Hilbert transform diverges almost everywhere.*

If we put $f(x) = (\log n)^{-1}$ $n < x \leqq n+1$ $(n = 1, 2, \cdots)$ and vanish elsewhere. This $f(x)$ proves this theorem. This example shows that the case $\alpha = 1$ is a critical case in some sense and if we wish to treat the Hilbert transform in the case $\alpha \geqq 1$, we should call a modified definition. These will be argued in other place.

References

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