134. On the Structure of the Associated Modular

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Let R be a modulared semi-ordered linear space[1] with a modular m. The structure of the conjugate modular $\overline{m}$ on the conjugate space $\overline{R}^m$ is investigated in detail.[2] On the other hand, it is known[3] that if the norm on R by m is not continuous, $\overline{R}^m$ constitutes a *proper* normal manifold of the associated space $\widetilde{R}^m$. In this short note, we shall determine completely the structure of the associated modular $\widetilde{m}$ on the orthogonal complement $(\overline{R}^m)^\perp$ of $\overline{R}^m$ in $\widetilde{R}^m$.

Theorem. *The associated modular $\widetilde{m}$ is linear[4] on $(\overline{R}^m)^\perp$; more precisely it is given by the formula:*

$$\widetilde{m}(\widetilde{a}) = \sup_{m(x)<\infty} |\widetilde{a}(x)| \qquad \textit{for all } \widetilde{a} \in (\overline{R}^m)^\perp.$$

Proof. There exists[5] a normal manifold N of R such that m is semi-simple on N and is singular on $N^\perp$. It is known that N is semi-regular[6] and the associated modular $\widetilde{m}$ is linear on $[N^\perp]\widetilde{R}^m$. Thus to prove Theorem we may assume that R is semi-regular.

Let $0 \leqq \widetilde{a} \in (\overline{R}^m)^\perp$ and $0 \leqq a \in R$ $m(a) < \infty$.

Put $$\boldsymbol{F} = \{x;\ 0 \leqq x \leqq a \quad \widetilde{a}(x) = 0\}.$$

Then it is a lattice manifold. Putting $e = \bigcup_{x \in \boldsymbol{F}} x$, we shall show first that $a = e$. For this purpose, it is sufficient to prove that

$$\overline{x}(a-e) \leqq \varepsilon \qquad \text{for any } 0 \leqq \overline{x} \in \overline{R}^m \text{ and } \varepsilon > 0,$$

because R is semi-regular by assumption. Since $\widetilde{a} \frown \overline{x} = 0$, there exist[7] $\{b_\nu\}_{\nu=1}^\infty \subset R$ such that

$$0 \leqq b_\nu \leqq a \quad \text{and} \quad \overline{x}(a-b_\nu) + \widetilde{a}(b_\nu) \leqq \varepsilon/2^\nu \quad (\nu = 1, 2, \cdots).$$

Putting $b = \bigcap_{\nu=1}^\infty b_\nu$, we have $0 \leqq \widetilde{a}(b) \leqq \inf_{\nu=1,2,\cdots} \widetilde{a}(b_\nu) = 0$, namely $b \in \boldsymbol{F}$. Further universal continuity of $\overline{x}$ implies $\overline{x}(a-b) = \overline{x}(\bigcup_{\nu=1}^\infty (a-b_\nu)) \leqq \sum_{\nu=1}^\infty \overline{x}(a-b_\nu) \leqq \varepsilon$. From this and the definition of e it follows that

1) We use the definitions, terminology, and notations in H. Nakano: Modulared Semi-ordered Linear Spaces, Maruzen, Tokyo (1950).

2) Ibid., §§ 41–46.

3) Ibid., Theorem 31.10.

4) $\widetilde{m}(\xi \widetilde{a}) = \xi \widetilde{m}(\widetilde{a})$ for all $\xi \geqq 0$.

5) Ibid., § 35.

6) Semi-regularity means that $\overline{x}(a) = 0$ (for all $\overline{x} \in \overline{R}^m$) implies $a = 0$.

7) Ibid., § 18.

$$\bar{x}(a-e) \leqq \bar{x}(a-b) \leqq \varepsilon.$$

Thus we have proved $a=e$.

Since $\sup_{x \in F} m(x) = m(a)$ by semi-continuity[8] of m, for any $\varepsilon > 0$ there exists $c \in \boldsymbol{F}$ such that

$$m(a-c) \leqq m(a) - m(c) \leqq \varepsilon.$$

Since
$$\tilde{m}(\tilde{a}) = \sup_{m(x)<\infty} \{\tilde{a}(x) - m(x)\}$$

by the definition of the associated modular, we obtain

$$\tilde{a}(a) = \tilde{a}(a-c) + \tilde{a}(c) = \tilde{a}(a-c)$$
$$\leqq \tilde{a}(a-c) - m(a-c) + \varepsilon \leqq \tilde{m}(\tilde{a}) + \varepsilon.$$

Since $0 \leqq a \in R$ $m(a) < \infty$ and $\varepsilon > 0$ are arbitrary, we can conclude

$$\sup_{m(x)<\infty} |\tilde{a}(x)| \leqq \tilde{m}(\tilde{a}).$$

Now the proof is complete, because the converse inequality is obviously valid.

Corollary. *The first norm and the second one by the associated modular* $\tilde{m}$ *coincide on* $(\bar{R}^m)^{\perp}$, *and*

$$\|\tilde{a} + \tilde{b}\| = \|\tilde{a}\| + \|\tilde{b}\| \qquad \text{for all } 0 \leqq \tilde{a}, \tilde{b} \in (\bar{R}^m)^{\perp}.$$

Remark. The assertion of Theorem is in essence a reformulation of *reflexivity of a semi-continuous modular.*[9]

8) Semi-continuity means that $0 \leqq x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $\sup_\lambda m(x_\lambda) = m(x)$.

9) See H. Nakano: Modulars on semi-ordered linear spaces I, Jour. Fac. Sci. Hokkaido Univ., ser. I, **13**, 41–52 (1956).