130. *On Linear Functionals of W^*-algebras*

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1. We shall explain the background of our study.

Let B be the W^*-algebra of all bounded operators on a Hilbert space H, then σ-weakly continuous linear functionals on B are identified with operators of trace class u in H as follows: $\psi_u(a)=\mathrm{Tr}(ua)$ $(a \in B)$. Self-adjoint (resp. positive) operators u of trace class correspond exactly to σ-weakly continuous self-adjoint (resp. positive) linear functionals ψ_u and the trace-norm $\|u\|_1=\mathrm{Tr}((u^*u)^{1/2})$ of u is equal to the norm $\|\psi_u\|$ of corresponding functionals. If u is self-adjoint, it can be written under $u=v-w$, where v and w are its positive and negative parts, and $\|u\|_1=\|v\|_1+\|w\|_1$. Besides, if we have $u=v'-w'$, where $v', w'\geqq 0$ and $\|u\|_1=\|v'\|_1+\|w'\|_1$, then we can easily show that $v=v'$ and $w=w'$. Namely: A σ-weakly continuous self-adjoint functional ψ_u on B can be written under $\psi_u=\psi_v-\psi_w$, where $\psi_v, \psi_w \geqq 0$ such that $\|\psi_u\|=\|\psi_v\|+\|\psi_w\|$, and such decomposition is unique. Grothendieck [3] has shown that this fact holds also valid in general W^*-algebras.

On the other hand, we know a stronger fact in B as follows: Let t be an operator of trace class, $t=v|t|$ $(|t|=(t^*t)^{1/2})$ its polar decomposition, then $\|t\|_1=\|\,|t|\,\|_1$ and v is a partially isometric operator $(\in B)$ having the range projection of $|t|$ as the initial projection. Now we consider the functional ψ_t, and denote $\psi_t(xy)=\hat{Y}\psi_t(x)$ and $\psi_t(yx)=\hat{\hat{Y}}\psi_t(x)$ for $x, y \in B$, then since $\psi_t(xy)=\mathrm{Tr}(txy)=\mathrm{Tr}(ytx)$, the above fact implies: $\psi_t=\hat{V}\psi_{|t|}$, $\|\psi_t\|=\|\psi_{|t|}\|$ and $\hat{V}$ is a partially isometric operator having the support $S(\psi_{|t|})$ of $\psi_{|t|}$ as the initial projection, where for $\psi\geqq 0$, $S(\psi)=I-\sup e$ [e, projections such that $\psi(e)=0$].

Moreover we can easily show that such decomposition is unique, and call this decomposition *the polar decomposition of functionals.*

Our purpose of this note is to show that the polar decomposition of functionals is also valid in general W^*-algebras.

2. We shall state

Theorem 1. *Suppose a W^*-algebra M realized as a W^*-subalgebra of the algebra B on a Hilbert space H, then a σ-weakly continuous linear functional ψ on M is the restriction of a σ-weakly continuous linear functional of the same norm on B.*

Proof. It is enough to suppose $\|\psi\|=1$. Let S be the unit sphere of M and $F=\{a \mid |\psi(a)|=1,\ a \in S\}$, then F is a non-void, convex, σ-

weakly compact set by the σ-weak compactness of S. Let a be an extreme point of F, then a is also extreme in S; put $u=\overline{\psi(a)}a$, then u is extreme in S and $\psi(u)=1$. By a theorem of Kadison [2], an extreme point u is a partially isometric operator such that $(I-uu^*)M(I-u^*u)=(0)$. Therefore by the theorem of comparability, there is a central projection z of M such that $I-uu^*\leqq z$ and $I-u^*u\leqq I-z$; hence $uu^*\geqq I-z$ and $u^*u\geqq z$, so that $u(I-z)u^*(I-z)=I-z$ and $u^*zuz=z$. Since $\psi(x)=\psi(xz)+\psi(x(I-z))$, if it is shown that the restrictions ψ_1 and ψ_2 of ψ on Mz and $M(I-z)$ are extendable to functionals of same norm on zBz and $(I-z)B(I-z)$ respectively, ψ is extendable to a functional $\tilde{\psi}$ of same norm on $zBz+(I-z)B(I-z)$; then, define $\tilde{\psi}(x)=0$ on $(I-z)Bz+zB(I-z)$, $\tilde{\psi}$ is extendable to a functional of same norm on B. Since $\psi_1(uz)=||\psi_1||$ and so $\hat{U}\hat{Z}\psi_1(I)=||\psi_1||$ and $||\hat{U}\psi_1||\leqq ||u||_\infty||\psi_1||=||\psi_1||$, where $||\cdot||_\infty$ is the uniform norm of an operator, by the well-known theorem $\hat{U}\hat{Z}\psi_1$ is positive; hence by the result of Dixmier [1] $\hat{U}\hat{Z}\psi_1$ is extendable to a positive normal functional ξ_1 of same norm on zBz. Then

$$\hat{Z}\hat{U}^*\xi_1(x)=\xi_1(xzu^*)=\hat{U}\hat{Z}\psi_1(xzu^*)=\psi_1(xzu^*uz)=\psi_1(xz)=\psi_1(x) \quad \text{for}$$

$x\in Mz$; hence $\hat{Z}\hat{U}^*\xi_1$ satisfies our demand. Analogously $(\widehat{I-Z})\hat{\hat{U}}\psi_2$ is positive and so it is extendable to a positive normal functional ξ_2 of same norm on $(I-z)B(I-z)$.

Then

$$\hat{\hat{U}}^*(\widehat{I-Z})\xi_2(x)=\xi_2((I-z)u^*z)=(\widehat{I-Z})\hat{\hat{U}}\psi_2((I-z)u^*x)$$
$$=\psi_2(u(I-z)(I-z)u^*x)=\psi_2(x) \quad \text{for } x\in M(I-z);$$

hence $\hat{\hat{U}}^*(\widehat{I-Z})\xi_2$ satisfies our demand, this completes the proof.

Theorem 2. *Let M be a W^*-algebra and ψ a σ-weakly continuous linear functional on M, then it can be written under $\psi=\hat{V}\varphi$, where φ is a positive normal functional, $||\psi||=||\varphi||$ and $\hat{V}$ is a partially isometric operator of M having the support $S(\varphi)$ of φ as the initial projection, where $S(\varphi)=I-\sup e$ [e, projection of M such that $\varphi(e)=0$]. Moreover, such decomposition is unique.*

We shall call the above φ *the absolute value* of ψ and denote it by $|\psi|$ [cf. 4].

Proof. It is enough to suppose $||\psi||=1$. Let u be a partially isometric operator of M such that $\psi(u)=1$, then $\hat{U}\psi$ is positive. Moreover, since $uu^*u=u$, $\psi(u)=\psi(uu^*u)=\hat{U}\psi(uu^*)=1$. Therefore $uu^*\geqq S(\hat{U}\psi)$. Put $w=u^*S(\hat{U}\psi)$, then $w^*w=S(\hat{U}\psi)$; hence w is a partially isometric operator having $S(\hat{U}\psi)$ as the initial projection. Moreover $\hat{U}\psi(x)=\hat{U}\psi(xS(\hat{U}\psi))=\psi(xS(\hat{U}\psi)u)=\psi(xw^*)=\hat{W}^*\psi(x)$ for all $x \varepsilon M$; hence $\hat{U}\psi = \hat{W}^*\psi$.

Now we show

Lemma. *Let p and q be projections such that $p = ww^*$ and $q = w^*w$, then $\psi(x) = \psi(xp)$ and $\psi(x) = \psi(qx)$ for all $x \varepsilon M$.*

Proof. We can suppose M realized as a W^*-subalgebra of B on a Hilbert space H. By Theorem 1, ψ is the restriction of a σ-weakly continuous linear functional $\tilde{\psi}$ of same norm on B. Then

$$\tilde{\psi}(w^*) = \psi(w^*) = 1 \text{ and } w^*p = w^*,\ qw^* = w^*.$$

Therefore, put $\tilde{\psi}(x) = \mathrm{Tr}(tx)$ (t: operator of trace class), then

$$\sup_{\|x\|_\infty \leqq 1, x \in Bp} |\tilde{\psi}(x)| = \sup_{\|x\|_\infty \leqq 1, x \in B} |\mathrm{Tr}(txp)| = \sup_{\|x\|_\infty \leqq 1, x \in B} |\mathrm{Tr}(ptx)| = \|pt\|_1 = \mathrm{Tr}((t^*pt)^{1/2})$$
$$= 1 = \|t\|_1 = \mathrm{Tr}((t^*t)^{1/2}).$$

On the other hand, $t^*pt \leqq t^*t$, so that $(t^*pt)^{1/2} \leqq (t^*t)^{1/2}$ [cf. 4]; hence by the above equality $(t^*pt)^{1/2} = (t^*t)^{1/2}$, $t^*pt = t^*t$ and $(I-p)t = 0$. Therefore $\tilde{\psi}(x(I-p)) = \mathrm{Tr}(tx(I-p)) = \mathrm{Tr}((I-p)tx) = 0$; hence $\psi(x) = \psi(xp)$ for all $x \varepsilon M$.

Analogously

$$\sup_{\|x\|_\infty \leqq 1, x \in qB} |\tilde{\psi}(x)| = \sup_{\|x\|_\infty \leqq 1, x \in B} |\mathrm{Tr}(tqx)| = \|tq\|_1 = \|qt^*\|_1 = \mathrm{Tr}((tqt^*)^{1/2}) = 1$$
$$= \mathrm{Tr}((t^*t)^{1/2}) = \|t\|_1 = \|t^*\|_1 = \mathrm{Tr}((tt^*)^{1/2});$$

hence $tqt^* = tt^*$ and so $(I-q)t^* = 0$, $t(I-q) = 0$. Therefore $\tilde{\psi}((I-q)x) = \mathrm{Tr}(t(I-q)x) = 0$; hence $\psi(x) = \psi(qx)$ for all $x \varepsilon M$.
This completes the proof.

By the above lemma,

$$\psi(x) = \psi(xp) = \psi(xww^*) = \hat{W}(\hat{W}^*\psi)(x) \quad \text{for all } x \varepsilon M;$$

hence $\hat{W}(\hat{W}^*\psi) = \psi$. Taking $\hat{W}^*\psi$ as φ and $\hat{W}$ as $\hat{V}$, the decomposition $\psi = \hat{V}\varphi = \hat{W}(\hat{W}^*\psi)$ satisfies the first part of Theorem.

Next we shall show the unicity. Suppose that ψ can be written under $\psi = \hat{V}'\varphi'$, where $\varphi' \geqq 0$, $\|\psi\| = \|\varphi'\|$ and $\hat{V}'$ is a partially isometric operator of M having $S(\varphi')$ as the initial projection. Since $\psi(v^*) = \psi(v'^*) = 1$, ψ is zero on $(I-v^*v)M$ and $(I-v'^*v')M$ by the same reason with Lemma; since $(I-v^*v)M + (I-v'^*v')M$ is a right ideal, its closure E is also a right ideal; hence by the well-known theorem of W^*-algebras there is a projection e of M such that $E = eM$. Assume that $I - v^*v < e$, then $v^*v > I - e$. Then, we have

$$1 = \psi(v^*) = \psi((I-e)v^*) = \hat{V}\varphi((I-e)v^*) = \varphi((I-e)v^*v) = \varphi(I-e).$$

This contradicts that v^*v is the support of φ; hence $I - e = v^*v$ and analogously $I - e = v'^*v'$; hence the final projection of v'^* is $S(\varphi)$, so that $S(\varphi)v'^* = v'^*$. On the other hand, by the same reason with Lemma ψ is zero on $M(I-vv^*)$ and $M(I-v'v'^*)$; hence its closure E' is a left ideal and so there is a projection e' of M such that $E' = Me'$. Assume that $I - vv^* < e'$, then $vv^* > I - e'$ and moreover

$$\psi(v^*)=\psi(v^*(I-e'))=\hat{V}\varphi(v^*(I-e'))=\varphi(v^*(I-e')v)=1.$$

On the other hand, since $I-e'<vv^*$, $v^*(I-e')v<v^*v=S(\varphi)$, this is a contradiction; hence $e'=I-vv^*$ and analogouly $e'=I-v'v'^*$, so that $vv^*=v'v'^*$. Then

$$\varphi(S(\varphi))=1=\psi(v'^*)=\hat{V}\hat{V}^*\psi(v'^*)=\hat{V}^*\psi(v'^*v).$$

Since $S(\varphi)v'^*vS(\varphi)=v'^*v=a+ib$ (a, b self-adjoint), then $a\leqq S(\varphi)$; hence by the above equality $a=S(\varphi)$; therefore by $\|v'^*v\|_\infty\leqq 1$, $b=0$, so that $v'^*v=S(\varphi)$. Therefore $v'v'^*v=vv^*v=v=v'S(\varphi)=v'v^*v=v'v'^*v'=v'$.

Therefore

$$\hat{V}\varphi(x)=\varphi(xv)=\varphi(xv')=\hat{V}'\varphi'(x)=\varphi'(xv') \quad \text{for all } x\in M.$$

Hence

$$\varphi(yv'^*v')=\varphi(y)=\varphi'(yv'^*v')=\varphi'(y) \quad \text{for } y\geqq 0 \text{ and } y\in S(\varphi)MS(\varphi).$$

Since φ and φ' have the same support $S(\varphi)$, the above equality implies $\varphi=\varphi'$.

This completes the proof.

References

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