5. On Approximation of Quasi-conformal Mapping

By Kêichi SHIBATA

(Comm. by K. KUNUGI, M.J.A., Jan. 12, 1959)

In this short note we are concerned with approximation to the general (not necessarily differentiable) quasi-conformal mapping by means of the smooth ones under the condition that the correspondence of a finite number of boundary points shall remain fixed.

In the course of our proof Ahlfors existence theorem plays an important rôle. The notations employed here for convenience are as follows:

$\mathfrak{F}$: The class of all the quasi-conformal mappings between the upper half-planes,

$M(z;\rho;g)$: Areal mean of an integrable function $g(z)$ over the disk $|\zeta-z|\leq\rho$, i.e.

$$M(z;\rho;g)=\frac{1}{\pi\rho^2}\int_0^\rho\int_0^{2\pi}g(z+re^{i\theta})rd\theta\,dr.$$

Proposition. Let $w=f(z)$ *be a quasi-conformal mapping in Pfluger-Ahlfors sense which is a homeomorphism between* $\Im z>0$ *and* $\Im w>0$. *Let* $x_1<x_2<\cdots<x_{k-1}<x_k$ *be points on* $\Im z=0$ *and* $f(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,k)$. *Then there exists a sequence* $\{f_n(z)\}$ *of quasi-conformal mappings* C^1 *between* $\Im z>0$ *and* $\Im w>0$, *such that* $f_n(z)$ *converges to* $f(z)$ *uniformly in* $\Im z>0$ *as* $n\to\infty$ *with the condition* $f_n(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,k)$ *and* $|\partial f_n/\partial z|$ *has a positive lower bound depending only on* n.

Proof. Mathematical induction with respect to the number of distinguished boundary points is available.

1) We first show that the proposition is true in case $k=3$. We may assume without loss of generality $x_1=u_1=-\infty$, $x_2=u_2=-1$, $x_3=u_3=0$.

Let $\{R_n\}$ and $\{\varepsilon_n\}$ be two sequences of positive numbers such that $R_n\uparrow\infty$ and $\varepsilon_n\downarrow 0$ respectively as $n\to\infty$. Let D_n be the domain which is the intersection of the disk $|z|<R_n$ and the half-plane $\Im z>\varepsilon_n$. We approximate the eccentricity $h(z)=\frac{\partial f}{\partial\bar z}\Big/\frac{\partial f}{\partial z}$ of the given mapping $w=f(z)$ by a sequence of functions $h_n(z)$ $(n=1,2,\cdots)$ which satisfies the following conditions:

i) $$h_n(z)=\begin{cases}h(z) & z\in D_n\\ 0 & |z|\geq R_n+1,\end{cases}$$

ii) $$h_n(\bar z)=\overline{h_n(z)},$$

iii) $h_n(z)$ fulfils the Hölder condition of order α $(0<\alpha\leq 1)$ for $|z|<\infty$,

iv) $$|h_n(z)| \le \sup |h(z)|.$$

According to Ahlfors theory [1] it is possible to construct the mapping function

$$\varphi_n(z) = z + \frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{g_n(\zeta)}{z-\zeta}\,d\xi d\eta \qquad (\zeta = \xi + i\eta)$$

which supplies a homeomorphism between the whole z- and w-plane. The construction of the integrand is carried out by making use of the Neumann series

$$q_n = h_n + h_n T h_n + h_n T h_n T h_n + \cdots,$$

where T denotes the operator of the Hilbert transform:

$$Tg(z) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{g(\zeta)}{(z-\zeta)^2}\,d\xi d\eta.$$

The function $w=\varphi_n(z)$ is symmetric with respect to the real axis. Set

$$f_n(z) = \frac{\varphi_n(z)-\varphi_n(0)}{\varphi_n(0)-\varphi_n(-1)}.$$

Then the function $f(z)$ belongs to $\mathfrak{F}$ and satisfies the condition $f_n(-\infty) = -\infty$, $f_n(-1)=-1$, $f_n(0)=0$. The sequence $\{f_n(z)\}$ converges towards $f(z)$ uniformly in $\Im z>0$ as $n\to\infty$, since the composite mapping $f_n\circ f^{-1}$ tends to the identity uniformly in the upper half-plane (cf. [2]). Moreover, in virtue of Ahlfors theorem [1] we have

$$\left|\frac{\partial f_n}{\partial z}\right| = \frac{1}{\varphi_n(0)-\varphi_n(-1)}\left|\frac{\partial \varphi_n}{\partial z}\right| \ge \delta_n > 0.$$

2) Suppose that the above proposition be true in case $k=m$. Our assumption is: There exists a sequence $\{f_{n,m}(z)\}$ of mappings belonging to $\mathfrak{F}$ convergent to $f(z)$ uniformly in $\Im z>0$, such that $f_{n,m}(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,m)$ and that $\left|\frac{\partial f_{n,m}}{\partial z}\right| \ge \delta_{n,m}>0$.

Set

$$\psi_n(z) = \begin{cases} f_{n,m}(z) & \Re z \le x_m \\ \dfrac{u_{m+1}-u_m}{f_{n,m}(x_{m+1})-u_m}[\Re f_{n,m}(z)-u_m] + u_m + i\Im f_{n,m}(z) & \Re z > x_m. \end{cases}$$

Then the mapping $w=\psi_n(z)$ is quasi-conformal and this function is continuously differentiable except on the straight line $\Re z = x_m$. Obviously we have $\psi_n(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,m,m+1)$ and

$$\left|\frac{\partial \psi_n}{\partial z}\right| \ge \begin{cases} \delta_{n,m}>0 & \Re z < x_m \\ \dfrac{1}{2}\left[\left|1+\dfrac{u_{m+1}-u_m}{f_{n,m}(x_{m+1})-u_m}\right|\cdot\left|\dfrac{\partial f_{n,m}}{\partial z}\right| - \left|1-\dfrac{u_{m+1}-u_m}{f_{n,m}(x_{m+1})-u_m}\right|\cdot\left|\dfrac{\partial f_{n,m}}{\partial \bar z}\right|\right] & \Re z > x_m. \end{cases}$$

Thus the quantity $\left|\frac{\partial \psi_n}{\partial z}\right|$ has a positive lower bound δ_n' depending on n except on $\Re z = x_m$. One sees by simple computations that there exists a positive number δ depending not on z but on n, such that $M(z;\rho;\psi_n)$ is locally one-to-one quasi-conformal so far as $\rho\le\delta$.

Therefore one can the function $\psi_n(z)$ smooth by averaging suitably in the neighbourhood of $\Re z=x_m$: We set with a positive number $a_n<\rho$

$$f_n^*(z)=f_{n,m+1}(z)=\begin{cases}\psi_n(z) & |\Re z-x_m|\geq 1\\ M(z;a(1-|\Re z-x_m|);\psi_n) & |\Re z-x_m|<1,\ \Im z\geq 1\\ M(z;a(1-|z-i-x_m|);\psi_n) & \\ \qquad |\Re z-x_m|<1,\ 0<\Im z<1,\ |z-i-x_m|\leq 1 & \\ \psi_n(z) \quad |\Re z-x_m|<1,\ 0<\Im z<1,\ |z-i-x_m|>1. & \end{cases}$$

Then $w=f^*(z)$ belongs to $\mathfrak{F}$ and that C^1, since the boundary correspondence is one-to-one. One can construct in this way the sequence $\{f_n^*(z)\}$ of quasi-conformal mappings between the upper half-planes, which converges to $f(z)$ uniformly in $\Im z>0$ with the condition $f_n^*(x_\nu)=u_\nu$ $(\nu=1,2,\cdots,m,m+1)$ and $\left|\frac{\partial f_n^*}{\partial z}\right|\geq\delta_n^*>0$, Q.E.D.

Thereof we immediately obtain

Theorem. Let $w=f(z)$ be a quasi-conformal mapping in Pfluger-Ahlfors sense, which supplies a homeomorphism between $|z|<1$ and $|w|<1$. Let $e^{i\theta_\nu}$ $(\nu=1,2,\cdots,k)$ be points on the boundary $|z|=1$ and $f(e^{i\theta_\nu})=e^{it_\nu}$. Then there exists a sequence $\{f_n(z)\}$ of quasi-conformal mapping C^1 between $|z|<1$ and $|w|<1$, such that $f_n(z)$ converges to $f(z)$ uniformly on $|z|\leq 1$ as $n\to\infty$ with the condition $f_n(e^{i\theta_\nu})=e^{it_\nu}$ $(\nu=1,2,\cdots,k)$.

References

[1] Ahlfors, L. V.,: Conformality with respect to Riemannian metrics, Ann. Acad. Sci. Fenn., A, I, **206**, 1–22 (1955).

[2] Shibata, K.,: Remarks on the sequence of quasi-conformal mappings, Proc. Japan Acad., **32**, 665–670 (1956).