18. *Representation of Some Topological Algebras. II*

By Shouro KASAHARA
Kobe University

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4. Idempotents of rank 1. This section is devoted to note several fundamental statements concerning the idempotents in an algebra, which we shall need in what follows.

LEMMA 1.[1] *Let p be an idempotent in an algebra E. If Ep (resp. pE) is a minimal left (resp. minimal right) ideal of E, then pEp is a division algebra.*

Proof. It will suffice to prove the lemma in the case of Ep under the assumption that $pEp\neq\{0\}$. Since p is an idempotent, p is the identity in the algebra pEp. Let x be a non-zero element in pEp; then Ex contains $px=x$, so that $Ex=Ep$ since Ep is a minimal left ideal. It follows that $pEx=pEp$, and hence we have $pEpx=pEp$. Therefore the element x has a left inverse in pEp.

LEMMA 2. *Let E be an algebra satisfying the condition* (ii),[2] *and let p be a non-zero idempotent in E. If pEp is a division algebra, then Ep is a minimal left ideal and pE is a minimal right ideal of E.*

Proof. Let I be a proper non-zero left ideal contained in Ep, and a be a non-zero element in I. Then by the condition (ii) we can find an element $u\in E$ such that $pua\neq 0$. Since pua is contained in the division algebra pEp, it has an inverse pxp in pEp; then $pxpua=p$. Therefore the left ideal I contains the element p, and so I coincides with Ep contrary to the assumption. Similarly we can prove that pE is a minimal right ideal.

LEMMA 3. *Let p be an idempotent in a Hausdorff topological algebra E, and A be a closed subset of E. If Ap (resp. pA) is contained in A, then the set Ap (resp. pA) is closed.*

Proof. It will suffice to show that Ap is closed, under the assumption that $Ap\subseteq A$. Let $\mathfrak{F}$ be a filter on the set Ap which converges to an element $a\in A$. Then, since each element of the filter $\mathfrak{F}$ is a subset of the set Ap, we have $\mathfrak{F}p=\{Bp;\ B\in\mathfrak{F}\}=\mathfrak{F}$. On the other hand, because of the continuity of the ring multiplication, the filter base $\mathfrak{F}p$ converges to ap, and so we have $a=ap$ since E is a Hausdorff space.

1) This lemma is essentially known, but we give a proof for the sake of completeness.

2) Cf. S. Kasahara: Representation of some topological algebras. I, Proc. Japan Acad., **34**, 355-360 (1958).

Let E be an algebra; we say that a non-zero element a of E is *of rank* 1 if, for any $x\in E$, there exists a number λ such that $axa=\lambda a$.

LEMMA 4. *Let E be an algebra satisfying the condition* (ii). *If p and q are non-zero elements in E of rank* 1, *then there exist a, b, c and $d\in E$ such that $p=aqb$ and $q=cpd$. If p and q are two non-zero idempotents in E of rank* 1, *then there exist $a, b\in E$ such that $p=aqb$ and $q=bpa$. Moreover, if p is a non-zero element in E of rank* 1, *then every non-zero element of the form xp or px is of rank* 1.

Proof. Let p and q be two non-zero elements in E of rank 1, and let $puq\neq 0$. Then by the condition (ii), we can find an element $s\in E$ such that $qspuq=q$. Thus $q=cpd$ for $c=qs$ and $d=uq$. It is now evident that there are two elements $a, b\in E$ such that $p=aqb$, but we will prove this by another way. Since p is of rank 1, we have $puqsp=\lambda p$ for some λ; but then $\lambda puq=puqspuq=puq$ and hence $\lambda=1$. Thus we have $p=puqsp$. Therefore if the elements p and q are idempotents, we can write $p=aqb$ and $q=bpa$ by taking $a=puq$ and $b=qsp$. Finally, if p is a non-zero element in E of rank 1, and if $xp\neq 0$, then we can find, for each $u\in E$, a number λ such that $puxp=\lambda p$. Therefore we have $(xp)u(xp)=\lambda xp$, and hence xp is of rank 1. The proof is similar for $px\neq 0$.

COROLLARY 1. *Let E be an algebra satisfying the condition* (ii). *If p and q are two non-zero elements in E of rank* 1, *then $EpE=EqE$.*

COROLLARY 2. *Let E be an algebra satisfying the condition* (ii) *and containing a non-zero element p of rank* 1. *Then every two-sided ideal $I\neq 0$ of E contains EpE. Consequently, the vector subspace spanned by the set EpE is the minimal two-sided ideal of E.*

Proof. Since $I\neq\{0\}$, I contains a non-zero element a. By the condition (ii) there exists an element $u\in E$ such that $aup\neq 0$, and so again by the condition (ii) we can find an element $v\in E$ such that $pvaup\neq 0$. Therefore, the two-sided ideal I contains the element $pvaup$ and so p. It follows that EpE is contained in I.

COROLLARY 3. *Let E be an algebra satisfying the condition* (ii). *If p and q are two non-zero elements in E of rank* 1, *then the vector space pEq is of one dimension.*

Proof. By Lemma 4, there exist two elements $a, b\in E$ such that $p=aqb$. Hence we have $pxq=aqbxq$ for every $x\in E$, and so the vector subspace pEq is spanned by aq.

LEMMA 5. *Let p and q be two non-zero idempotents of rank* 1 *in a commutative algebra. Then $p\neq q$ if and only if $pq=0$.*

Proof. If $pq\neq 0$, then we have $\mu p=qp=pq=\lambda q$ for some non-zero numbers λ and μ. Therefore $\frac{\mu}{\lambda}p=q=q^2=\left(\frac{\mu}{\lambda}\right)^2p$ and so $p=q$. Conversely, if $p=q$, then $pq=p^2=p\neq 0$.

5. Uniqueness of the representations. In this section we show that each vector space mentioned in Theorems 5, 7, and 8 is unique to an isomorphism.

THEOREM 9. *Let E be an algebra satisfying the condition* (ii). *If p and q are non-zero idempotents in E of rank* 1, *then the vector space Ep (resp. pE) is isomorphic with the vector space Eq (resp. qE). Let E be a Hausdorff topological algebra satisfying the condition* (ii) *and containing two non-zero idempotents p, q of rank* 1. *Let $\mathfrak{A}$ (resp. $\mathfrak{A}'$) be the family of all left bounded sets contained in Ep (resp. in Eq), and $\mathfrak{B}$ (resp. $\mathfrak{B}'$) be the family of all right bounded sets contained in pE (resp. in qE). Then the vector space Ep with the right $\mathfrak{B}$-topology (resp. pE with the left $\mathfrak{A}$-topology) and the vector space Eq with the right $\mathfrak{B}'$-topology (resp. qE with the left $\mathfrak{A}'$-topology) are isomorphic (topologically and algebraically).*

Proof. By Lemma 4 of the section 4, we can find two elements $a, b\in E$ such that $p=aqb$ and $q=bpa$. We define a linear mapping φ of Ep into Eq by $\varphi(xp)=xpaq$. Then φ is an isomorphism of Ep onto Eq. In fact, if $xp\neq yp$, we have, by the condition (ii), $xpwq\neq ypwq$ for some $w\in E$, and so $xpaqbwq\neq ypaqbwq$; this implies that $xpaq\neq ypaq$, namely that $\varphi(xp)\neq\varphi(yp)$. Moreover, since $\varphi(xqbp)=x(qbpaq)=xq$ for all $x\in E$, we see that φ maps Ep onto Eq. Now the linear mapping ψ defined by $\psi(qx)=paqx$ is also an isomorphism of qE onto pE. In fact, if $qx\neq qy$, then we have, for some $w\in E$, $pwqx\neq pwqy$, that is $pwbpaqx\neq pwbpaqy$, and so $paqx\neq paqy$. This shows that the mapping ψ is one-to-one. On the other hand, since $\psi(qbpx)=paqbpx=px$ for every $x\in E$, ψ is onto. Thus the vector spaces pE and qE are isomorphic. Let us suppose now that the algebra E is equipped with a topology compatible with the structure of E. Then with the notation in the theorem, we have $\varphi(\mathfrak{A})=\mathfrak{A}'$ and $\psi(\mathfrak{B}')=\mathfrak{B}$. In fact, if $A\in\mathfrak{A}$, then $Aaq=\varphi(A)$ is left bounded in Eq, that is $\varphi(A)\in\mathfrak{A}'$; if $A\in\mathfrak{A}'$, then $Abp\in\mathfrak{A}$ and $\varphi(Abp)=Abpaq=A$ since $A\subseteq Eq$. By an analogous way, we can prove that $\psi(\mathfrak{B}')=\mathfrak{B}$. As was pointed out in the section 3, the vector spaces Ep and pE constitute a separated dual system, and of course Eq and qE too. Therefore, in order to complete the proof of the theorem, it is sufficient to show that the conjugate mapping of φ is ψ, or equivalently that $qyxpaq=\lambda q$ if and only if $paqyxp=\lambda p$ $(x, y\in E)$. For this, it suffices to prove that $qyxpaq=\lambda q$ and $paqyxp=\mu p$ imply $\lambda=\mu$. But this is obvious by the following simple computation:

$$\lambda paq=pa(qyxpaq)=(paqyxp)aq=\mu paq.$$

THEOREM 10. *Let X be a locally convex Hausdorff vector space,* *and $\mathfrak{S}$ be a covering of X consisting of closed bounded convex and circled sets. Then each subalgebra E of the algebra $\mathcal{L}_{\mathfrak{S}}(X, X)$ containing all continuous linear mappings of finite rank satisfies the conditions* (i′), (ii), (iii) *and* (iv).

Proof. Let z be an element of X' and z' be an element of the dual space X' of X. We shall denote by $z'\otimes z$ the linear mapping of X into itself: $x\to\langle x, z'\rangle z$. To verify the condition (ii), take two non-zero elements u, v in E. We can find then two points $x, y\in X$ such that $u(x)\neq 0$ and $v(y)\neq 0$. Since X is a Hausdorff space there exists an $x'\in X'$ such that $\langle v(y), x'\rangle\neq 0$. But then, as can easily be seen, the mapping $w=x'\otimes x$ possesses the required property. Now let $z\in X$ and $z'\in X'$ be such that $\langle z, z'\rangle=1$; then we have

$$(z'\otimes z)u(z'\otimes z)=z'\otimes z\circ z'\otimes u(z)=\langle u(z), z'\rangle z'\otimes z \quad \text{for every } u\in E$$

and
$$(z'\otimes z)(z'\otimes z)=z'\otimes z.$$

This proves that the condition (i′) is satisfied by taking $p=z'\otimes z$. Therefore, Ep is the set of all mappings of the form: $z'\otimes u(z)$, $u\in E$, and pE is the set of all mappings of the form: $u'z'\otimes z$, $u\in E$, where u' is the conjugate mapping of u. Since E contains all continuous linear mappings of finite rank, Ep and pE are isomorphic (algebraically) with X and with X' respectively. We shall now show that every right bounded set $B'\otimes z$ in pE is relatively compact in pE for the left $\{\{x\}; x\in Ep\}$-topology. To prove this, it is sufficient to show that the set B' is relatively compact for the weak topology $\sigma(X', X)$, or equivalently that the polar set $B'^{\circ}=\{x\in X; |\langle x, x'\rangle|\leqq 1 \text{ for every } x'\in B'\}$ of the set B' is a neighbourhood of 0 in X. Let us denote by V_0 a neighbourhood of 0 in X with the property that $\lambda z\in V_0$ if and only if $|\lambda|\leqq 1$. Then, since $B'\otimes z$ is right bounded, for any $A\in\mathfrak{S}$, there exist a member B of $\mathfrak{S}$ and an open neighbourhood U of 0 in X such that $(B'\otimes z)W(B, U)\subseteq W(A, V_0)$. In other words, for each $u\in W(B, U)$, the set $\langle u(A), B'\rangle z$ is contained in the neighbourhood V_0, and hence we have $u(A)\subseteq B'^{\circ}$ for every $u\in W(B, U)$. It follows that $W(B, U)$ is contained in the set $W(A, B'^{\circ})$. Now we can suppose, without loss of generality, that the set A is contained in B, because we have $W(B, U)\supseteq W(C, U)$ for the closed convex circled hull C of the set $A\smile B$. Let a be a point of A. Then there exists a positive number μ such that $\mu a\in B$ and that $\lambda>\mu$ implies $\lambda a\bar{\in} B$. Take a number α greater than μ. If there is an $x_0\in U$ which does not belong to $\alpha B'^{\circ}$, then U being open, we can find an $\eta>1$ such that $\eta x_0\in U$. Since $\mu\eta>\mu$, the point $\mu\eta a$ does not belong to B. Hence, by virtue of the Hahn-Banach theorem, there exists a continuous linear functional $x'\in X'$ such that $\langle\mu\eta a, x'\rangle=1$ and $|\langle b, x'\rangle|<1$ for all $b\in B$. It is easy to see that the mapping $u=\eta x'\otimes x_0$ belongs to the set $W(B, U)$ and that $u(\mu a)=x_0$. Since μa is a point of αA, this contradicts the assumption that $x_0\bar{\in}\alpha B'^{\circ}$. Therefore $(1/\alpha)U$

is contained in B'°, and so B'° is a neighbourhood of 0 in X. Thus the algebra E satisfies the condition (iii). We will now proceed to prove the condition (iv). Since the $(\mathfrak{A}, \mathfrak{B})$-topology is coaser than the original topology of E, it suffices to show that every neighbourhood $W(A, U)$ of 0 in E contains a neighbourhood of 0 for the $(\mathfrak{A}, \mathfrak{B})$-topology. Let $B \in \mathfrak{S}$, then for some $\lambda > 0$ we have $|\langle b, z'\rangle| \leqq \lambda$ for all $b \in B$. Therefore, for any neighbourhood V of 0 in X we have

$$W(\lambda A, V)(z' \otimes A) \subseteq W(B, V),$$

so that $z' \otimes A$ is left bounded in Ep. On the other hand, for any neighbourhood V of 0 in X, we can find a positive number α such that $\lambda z \in V$ whenever $|\lambda| \leqq \alpha$. But then we have

$$(U^{\circ} \otimes z) W(A, \alpha U) \subseteq W(A, V),$$

proving that $U^{\circ} \otimes z$ is right bounded in pE. Consequently, the set $W = W_l(z' \otimes A, W_r(U^{\circ} \otimes z, W(z, V_0)))$ is a neighbourhood of 0 for the $(\mathfrak{A}, \mathfrak{B})$-topology. Let $u \in W$; then since $(U^{\circ} \otimes z)(z' \otimes u(A)) = \langle u(A), U^{\circ}\rangle z' \otimes z$, we have $\langle u(A), U^{\circ}\rangle \langle z, z'\rangle z = \langle u(A), U^{\circ}\rangle z \subseteq V_0$, and hence $u(A) \subseteq U$. It follows that W is contained in $W(A, U)$. This completes the proof of the theorem.

As immediate consequences of the above two theorems, we have the following corollaries:

COROLLARY 1. *If E is an algebra isomorphic with a subalgebra of $\mathcal{L}(X, X)$ containing all continuous linear mappings of finite rank, then the vector space X is unique to an (algebraic) isomorphism.*

COROLLARY 2. *If E is a topological algebra isomorphic (topologically) with a subalgebra of $\mathcal{L}_{\mathfrak{S}}(X, X)$ containing all continuous linear mappings of finite rank, then the locally convex Hausdorff vector space X is unique to a (topological) isomorphism.*

COROLLARY 3. *Let X and Y be two locally convex Hausdorff vector spaces. If a subalgebra of $\mathcal{L}_{\mathfrak{S}}(X, X)$ containing all continuous linear mappings of finite rank is isomorphic with a subalgebra of $\mathcal{L}_{\mathfrak{T}}(Y, Y)$ containing all continuous linear mappings of finite rank, then X and Y are isomorphic.*

6. A remark on simple algebras. An algebra E is said to be *simple* if it contains no two-sided ideal other than $\{0\}$ and E. If a topological algebra E contains no closed two-sided ideal other than $\{0\}$ and E, then we call E *topological simple algebra*. Let X be a locally convex Hausdorff vector space; it is an immediate consequence of Corollary 2 of Lemma 4 of the section 4 and Theorem 10 that every two-sided ideal of the algebra $\mathcal{L}(X, X)$ contains all continuous linear mappings of finite rank. Therefore, if, for example, X is an infinite dimensional Banach space, then the algebra $\mathcal{L}(X, X)$ with the topology of uniform convergence on the unit sphere is not simple though it satisfies the condition (ii). The purpose of this section is to show that some simple algebras satisfy the condition (ii).[3]

LEMMA 1. *If E is a simple algebra, then either $EE=\{0\}$ and E is of one dimension, or it satisfies the following condition:*

$(*)$ *For any non-zero element $a\in E$, we can find two elements $x, y\in E$ such that $xa\neq 0$ and $ay\neq 0$.*

The same conclusion holds too for a Hausdorff topological simple algebra E.

Proof. Suppose for example that there exists a non-zero element a in the simple algebra E such that $xa=0$ for every $x\in E$. Then since $xaE=\{0\}$ for any $x\in E$, aE is a two-sided ideal of the algebra E, and hence $aE=E$ or $aE=\{0\}$ by the simplicity of E. But the first case does not occur. For if $aE=E$, then since $a^2=0$, we have $aE=\{0\}$, and so $E=\{0\}$, which contradicts $a\neq 0$. Therefore $aE=\{0\}$. Hence the two-sided ideal generated by the single element a is nothing more than the subspace spanned by a. But then since E is simple and $a\neq 0$, this ideal is E itself. To prove the case where E is a Hausdorff topological simple algebra, it will suffice to show that $a\neq 0$, $Ea=\{0\}$ and $\overline{aE}=E$ imply a contradiction, because every one dimensional subspace of E is closed. Now since $a\cdot aE=0\cdot E=\{0\}$, we have $a\cdot\overline{aE}=\{0\}$ by the continuity of the ring multiplication. Thus we have $E=\{0\}$, a contradiction.

LEMMA 2. *If E is either a simple algebra or a Hausdorff topological algebra and if $EE\neq\{0\}$, then the algebra E satisfies the condition* (ii).

Proof. For a contradiction, we suppose that the simple algebra E does not satisfy the condition (ii). Then there exist two non-zero elements $u, v\in E$ such that $uxv=0$ for every $x\in E$. It is clear that the subspace F spanned by the set EuE is a two-sided ideal of E. Since we can find, by Lemma 1, two elements $a, b\in E$ such that $aub\neq 0$, the two-sided ideal F coincides with E. Therefore, for any $x\in E$, we can find a finite number of elements $x_i, y_i\in E$ such that $x=\sum_{i=1}^{n} x_i u y_i$ and consequently we have $xv=\sum x_i u y_i v=\sum x_i 0=0$ for every $x\in E$; but this is impossible in view of Lemma 1. If E is a Hausdorff topological simple algebra, then the two-sided ideal F is dense in E, and hence we have also $Ev=\{0\}$, since $Fv=\{0\}$ by the assumption, and since the ring multiplication is continuous.

It is easy to see that the condition (ii) implies the condition $(*)$. But in general, the condition $(*)$ does not imply the condition (ii) as the following example shows: Let S be a set consisting of two points, say t_1 and t_2; then the algebra $\mathscr{F}(S, R)$ of all real valued functions defined on S satisfies the condition $(*)$ as can be readily seen. Let u and v be two functions on S such that $u(t_1)\neq 0\neq v(t_2)$ and $u(t_2)=v(t_1)=0$. Then we can not find any function x on S for which we have $uxv\neq 0$. Thus the algebra $\mathscr{F}(S, R)$ does not satisfy the condition (ii).

3) We can generalize the following lemmas to semi-groups or mobs.