13. *On Fejér Kernels*

By Kenji YANO

Department of Mathematics, Nara Women's University, Nara, Japan

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This note is a selection slightly modified of the parts concerning the properties of Fejér kernels and their conjugates studied in Yano [1-3], the paper [1] being written in Japanese. The results will improve the lemmas used in Gergen [4] and others.

1. Fejér kernels. The results in this article have been known classically in alternative forms. Here we shall deal with Fejér kernels and their conjugates at the same time. The n-th Fejér kernel of order α, $\alpha>-1$, is

$$\mathrm{K}_n^\alpha(t)=\frac{1}{2}+\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^\alpha \cos \nu t, \tag{1.1}$$

where A_n^γ, $-\infty<\gamma<\infty$, is defined by the identity

$$(1-x)^{-\gamma-1}=\sum_{n=0}^{\infty} A_n^\gamma x^n \quad (|x|<1), \tag{1.2*}$$

and its conjugate is

$$\overline{\mathrm{K}}_n^\alpha(t)=\frac{1}{A_n^\alpha}\sum_{\nu=1}^{n} A_{n-\nu}^\alpha \sin \nu t. \tag{1.3}$$

Putting

$$g_n^\alpha(t)=\mathrm{K}_n^\alpha(t)+i\overline{\mathrm{K}}_n^\alpha(t), \tag{1.4}$$

we have by (1.1) and (1.3)

$$g_n^\alpha(t)=-\frac{1}{2}+\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^\alpha e^{i\nu t}=-\frac{1}{2}+\frac{e^{int}}{A_n^\alpha}\sum_{\nu=0}^{n} A_\nu^\alpha e^{-i\nu t}.$$

Applying Abel's transformation to the last sum $\sum_{\nu=0}^{n}$ once,

$$g_n^\alpha(t)=\frac{i}{2}\cot\frac{1}{2}t+\frac{e^{int}}{A_n^\alpha(1-e^{-it})}\sum_{\nu=0}^{n} A_\nu^{\alpha-1}e^{-i\nu t}.$$

Generally, applying Abel's transformation to the last sum m-times successively we get

$$\begin{aligned} g_n^\alpha(t)=\frac{i}{2}\cot\frac{1}{2}t-\sum_{j=1}^{m}\frac{A_n^{\alpha-j}e^{-it}}{A_n^\alpha(1-e^{-it})^{j+1}} \\ +\frac{e^{int}}{A_n^\alpha(1-e^{-it})^{m+1}}\sum_{\nu=0}^{n} A_\nu^{\alpha-m-1}e^{-i\nu t}. \end{aligned} \tag{1.5}$$

When $m\geqq\alpha$ the series $\sum A_\nu^{\alpha-m-1}e^{-i\nu t}$ converges absolutely, and so by (1.2) and the foot-note *),

*) If in (1.2) x is a complex variable we consider of course the principal value of $(1-x)^{-\gamma-1}$ only, and this identity then holds also when $|x|=1$ provided that $\gamma\leqq-1$.

$$\sum_{\nu=0}^{n} A_\nu^{\alpha-m-1} e^{-i\nu t} = \sum_{\nu=0}^{\infty} - \sum_{\nu=n+1}^{\infty}$$
$$= (1-e^{-it})^{m-\alpha} - \sum_{\nu=n+1}^{\infty} A_\nu^{\alpha-m-1} e^{-i\nu t}.$$

Substituting this into (1.5) we get

$$(1.6)\qquad \begin{aligned} g_n^\alpha(t) = \frac{i}{2}\cot\frac{1}{2}t + \frac{e^{int}}{A_n^\alpha(1-e^{-it})^{\alpha+1}} - \sum_{j=1}^{m} \frac{A_n^{\alpha-j} e^{-it}}{A_n^\alpha(1-e^{-it})^{j+1}} \\ - \frac{1}{A_n^\alpha(1-e^{-it})^{m+1}} \sum_{\nu=n+1}^{\infty} A_\nu^{\alpha-m-1} e^{-i(\nu-n)t}, \end{aligned}$$

where, if $\alpha=0$ the last two terms vanish, and if $\alpha=m\geqq 1$ the last term only does. Moreover the last term is $O(1/n^m t^{m+1})$, and clearly for $n^{-1}\leqq t\leqq\pi$ the μ-th derivative of this term is

$$O(1/n^{m-\mu}t^{m+1}) \quad \text{for} \quad 1\leqq\mu\leqq[m-\alpha],$$

provided that $m\geqq\alpha+1$. Concerning the formula (1.6), cf. Zygmund [5, pp. 48, 184, 258–259, etc.]. From (1.6) and (1.4) we have

$$(1.7)\qquad \begin{aligned} \mathrm{K}_n^\alpha(t) &= \frac{\cos(nt+(\alpha+1)(t-\pi)/2)}{A_n^\alpha(2\sin(t/2))^{\alpha+1}} - \Bigg[\sum_{j=1}^{m} \frac{A_n^{\alpha-j}\cos(t-(j+1)(t-\pi)/2)}{A_n^\alpha(2\sin(t/2))^{j+1}} \\ &\quad + \frac{1}{A_n^\alpha(2\sin(t/2))^{m+1}} \sum_{\nu=n+1}^{\infty} A_\nu^{\alpha-m-1}\cos\Big((\nu-n)t-\frac{1}{2}(m+1)(t-\pi)\Big)\Bigg] \\ &= \Lambda_n^\alpha(t) + \mathrm{R}_n^\alpha(t), \end{aligned}$$

and

$$(1.8)\qquad \begin{aligned} \overline{\mathrm{K}}_n^\alpha(t) &= \frac{1}{2}\cot\frac{1}{2}t + \frac{\sin(nt+(\alpha+1)(t-\pi)/2)}{A_n^\alpha(2\sin(t/2))^{\alpha+1}} \\ &\quad + \Bigg[\sum_{j=1}^{m} \frac{A_n^{\alpha-j}\sin(t-(j+1)(t-\pi)/2)}{A_n^\alpha(2\sin(t/2))^{j+1}} \\ &\quad + \frac{1}{A_n^\alpha(2\sin(t/2))^{m+1}} \sum_{\nu=n+1}^{\infty} A_\nu^{\alpha-m-1}\sin\Big((\nu-n)t-\frac{1}{2}(m+1)(t-\pi)\Big)\Bigg] \\ &= \frac{1}{2}\cot\frac{1}{2}t + \overline{\Lambda}_n^\alpha(t) + \overline{\mathrm{R}}_n^\alpha(t). \end{aligned}$$

Remark 1. In the formula (1.8) the term corresponding to $j=1$ in the sum $\sum_{j=1}^{m}$ vanishes, but it is not the same in (1.7).

From (1.7) we can easily deduce Lemma 6 in Gergen [4]. Thus we have for $n^{-1}\leqq t\leqq\pi$ and $\mu=0,1,\cdots,[m-\alpha]$,

$$(1.9)\qquad \Big(\frac{d}{dt}\Big)^\mu \mathrm{R}_n^\alpha(t) = \begin{cases} O(1/n^{\alpha-\mu}t^{\alpha+1}) & (-1<\alpha\leqq 1) \\ O(1/nt^{\mu+2}) & (\alpha>-1), \end{cases}$$

$$(1.10)\qquad \Big(\frac{d}{dt}\Big)^\mu \overline{\mathrm{R}}_n^\alpha(t) = \begin{cases} O(1/n^{\alpha-\mu}t^{\alpha+1}) & (-1<\alpha\leqq 2) \\ O(1/n^2t^{\mu+3}) & (\alpha>-1). \end{cases}$$

Further, writing

$$D_n^\alpha(t) = \frac{1}{2}A_n^\alpha + \sum_{\nu=1}^{n} A_{n-\nu}^\alpha \cos\nu t = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} D_\nu(t),$$

where $D_n(t)=D_n^0(t)$ is the n-th Dirichlet kernel, we have

$$D_n^\alpha(t) = A_n^\alpha \mathrm{K}_n^\alpha(t),$$

which gives for $D_n^\alpha(t)$ the expression (1.7) multiplied by A_n^α, and in this case the number α may be quite arbitrary so far as it is real. It is analogous to $\overline{D}_n^\alpha(t)$, the conjugate to $D_n^\alpha(t)$.

2. Cesàro means of Fourier series and allied Fourier series. Let $f(t)$ be integrable in $(0, 2\pi)$, periodic with period 2π, and its Fourier series be

$$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n \cos nt+b_n \sin nt). \tag{2.1}$$

Then its allied Fourier series is

$$\sum_{n=1}^{\infty}(b_n \cos nt-a_n \sin nt). \tag{2.2}$$

We write, for fixed x

$$\varphi(t)=\varphi_x(t)=\frac{1}{2}[f(x+t)+f(x-t)],$$

$$\psi(t)=\psi_x(t)=\frac{1}{2}[f(x+t)-f(x-t)],$$

and let $\sigma_n^\alpha(x)$ where $\alpha>-1$, be the n-th Cesàro mean of order α of (2.1) at $t=x$, and $\overline{\sigma}_n^\alpha(x)$ be its conjugate. Then, as it is well known we have

$$\sigma_n^\alpha(x)=\frac{2}{\pi}\int_0^\pi \varphi(t)\,\mathrm{K}_n^\alpha(t)\,dt, \tag{2.3}$$

$$\overline{\sigma}_n^\alpha(x)=\frac{2}{\pi}\int_0^\pi \psi(t)\,\overline{\mathrm{K}}_n^\alpha(t)\,dt. \tag{2.4}$$

Precedingly, the author [2] proved that using (1.3),

$$\int_0^\pi \overline{\mathrm{K}}_n^\alpha(t)\,dt=\lambda(\alpha, n)+\log 2+o(1) \qquad (n\to\infty),$$

where

$$\lambda(\alpha, n)=\frac{1}{\alpha+1}+\frac{1}{\alpha+2}+\cdots+\frac{1}{\alpha+n}. \tag{2.5}$$

And more precisely

$$\int_0^\pi \overline{\mathrm{K}}_n^\alpha(t)\,dt=\log\frac{\alpha+n}{\alpha+1}+\log 2+c_\alpha+\begin{cases}O(1/n) & (\alpha\geqq 0)\\ O(1/n^{\alpha+1}) & (-1<\alpha<0),\end{cases}$$

where c_α is a constant depending on α only and $0<c_\alpha<1/(\alpha+1)$. Hence, for a function $f(t)$ which has a jump $2l$ at $t=x$ we have the following

THEOREM 1. *If $\alpha>-1$, and $\lambda(\alpha, n)$ is defined by (2.5), then*

$$\overline{\sigma}_n^\alpha(x)=\frac{2}{\pi}\int_0^\pi [\psi(t)-l]\overline{\mathrm{K}}_n^\alpha(t)\,dt$$
$$+\frac{2}{\pi}l[\lambda(\alpha, n)+\log 2]+o(1) \qquad (n\to\infty).$$

In the study of summability (C, α), $\alpha>-1$, of the Fourier series (2.1) at $t=x$, if we use the formula (2.3) itself then generally we must deal with two cases $-1<\alpha\leqq 1$ and $\alpha>1$ separately. This is a result from the behavior of $\mathrm{R}_n^\alpha(t)$ in (1.7). In order to get rid of this inconvenience, from the identity (1.6) the author [3] has deduced the following quasi Fejér kernels $L_n^\alpha(t)=L_n^\alpha(m,t)$, $m\geqq 1$, instead of $\mathrm{K}_n^\alpha(t)=L_n^\alpha(0,t)$.

THEOREM 2. Suppose that $\alpha>-1$, and

$$L_n^\alpha(t)=\alpha A_m^\alpha \sum_{k=0}^{m}(-1)^k\binom{m}{k}(\alpha+k)^{-1}\mathrm{K}_{n-k}^{\alpha+k}(t), \qquad (n\geqq m), \tag{2.6}$$

where m is a fixed positive integer arbitrarily chosen. Then, (I) a necessary and sufficient condition for

$$\sigma_n^\alpha(x)\equiv\frac{2}{\pi}\int_0^\pi \varphi(t)\,\mathrm{K}_n^\alpha(t)\,dt=s+o(n^{-\delta}) \qquad (n\to\infty),$$

where $s=s(x)$ and $\delta<1$, is that

$$\frac{2}{\pi}\int_0^\pi \varphi(t)\,L_n^\alpha(t)\,dt=s+o(n^{-\delta}) \qquad (n\to\infty).$$

(II) $L_n^\alpha(t)$ possess the following properties:

$$\frac{2}{\pi}\int_0^\pi L_n^\alpha(t)\,dt=1 \qquad (n\geqq m) \tag{2.7}$$

$$\left(\frac{d}{dt}\right)^\mu L_n^\alpha(t)=O(n^{\mu+1}) \qquad \text{for all } t \text{ and } \mu\geqq 0, \tag{2.8}$$

$$\left(\frac{d}{dt}\right)^\mu L_n^\alpha(t)=O(1/n^{\alpha-\mu}t^{\alpha+1}) \qquad \text{for } n^{-1}\leqq t\leqq\pi, \tag{2.9}$$

and $\mu=0,1,\cdots,[m-\alpha]$,

$$\int_t^\pi L_n^\alpha(u)\,du=O(1/n^{\alpha+1}t^{\alpha+1}) \qquad \text{for } n^{-1}\leqq t\leqq\pi, \tag{2.10}$$

as $n\to\infty$.

Observing that the number m may be as large as we wish, we may roughly say that throughout the value of $\alpha>-1$, the kernel $L_n^\alpha(t)=L_n^\alpha(m,t)$ with large m is, in a sense, equivalent to

$$\mathrm{K}_n^\alpha(t) \qquad \text{for } 0\leqq t\leqq n^{-1},$$

and to

$$\Lambda_n^\alpha(t) \qquad \text{for } n^{-1}\leqq t\leqq\pi,$$

using the notations in (1.7).

This theorem with $\delta=0$ is Theorem 4 in the paper [3], and the general case $\delta<1$ is proved implicitely in there.

Similarly we have the following theorem, in which the range of δ may be taken as $\delta<2$ by Remark 1.

THEOREM 3. Suppose that $\alpha>-1$, and

$$\overline{L}_n^\alpha(t)=\frac{(\alpha-1)\alpha}{m+1}A_m^\alpha\sum_{k=0}^{m}(-1)^k\binom{m}{k}((\alpha-1+k)(\alpha+k))^{-1}\overline{\mathrm{K}}_{n-k}^{\alpha+k}(t) \qquad (n\geqq m),$$

where m is a fixed positive integer arbitrarily chosen. Then, (I) a necessary and sufficient condition for

(2.11) $$\bar{\sigma}_n^\alpha(x) \equiv \frac{2}{\pi} \int_0^\pi \psi(t)\, \overline{\mathrm{K}}_n^\alpha(t)\, dt = \bar{s} + o(n^{-\delta}) \qquad (n \to \infty),$$

where $\bar{s} = \bar{s}(x)$ and $\delta < 2$, is that

(2.12) $$\frac{2}{\pi} \int_0^\pi \psi(t) \bar{L}_n^\alpha(t)\, dt = \bar{s} + o(n^{-\delta}) \qquad (n \to \infty).$$

(II) $\bar{L}_n^\alpha(t) = \bar{L}_n^\alpha(m, t)$ possess the following properties:

$$\left(\frac{d}{dt}\right)^\mu \bar{L}_n^\alpha(t) = O(n^{\mu+1}) \quad \text{for all } t \text{ and } \mu \geqq 0,$$

$$\left(\frac{d}{dt}\right)^\mu \left(\bar{L}_n^\alpha(t) - \frac{1}{2}\cot\frac{1}{2}t\right) = O(1/n^{\alpha-\mu} t^{\alpha+1})$$

$$\text{for } n^{-1} \leqq t \leqq \pi \text{ and } \mu = 0, 1, \cdots, [m-\alpha],$$

$$\int_t^\pi \left(\bar{L}_n^\alpha(u) - \frac{1}{2}\cot\frac{1}{2}u\right) du = O(1/n^{\alpha+1} t^{\alpha+1}) \quad \text{for } n^{-1} \leqq t \leqq \pi,$$

as $n \to \infty$.

The proof runs quite analogously as Theorem 2 using the following lemma in place of Lemma 3 in the paper [3].

LEMMA. If $\alpha > -1$, and σ_n^γ is the n-th Cesàro mean of a sequence $\{s_n;\ n = 0, 1, \cdots\}$, then a necessary and sufficient condition for

$$\sigma_n^\alpha = s + o(n^{-\delta}), \quad \delta < 2, \qquad (n \to \infty),$$

is that

$$\frac{(\alpha-1)\alpha}{m+1} A_m^\alpha \sum_{k=0}^m (-)^k \binom{m}{k} ((\alpha-1+k)(\alpha+k))^{-1} \sigma_{n-k}^{\alpha+k} = s + o(n^{-\delta}),$$

as $n \to \infty$, where m is a fixed positive integer arbitrarily chosen.

This lemma may be proved by the same argument as in the proof of Lemma 3 in [3], and we omit it.

Remark 2. Clearly, Theorem 3 holds when $\psi(t)$ is replaced by any other function integrable in $(0, \pi)$, e.g. $\theta(t) = \psi(t) - l$, where $l = l(x)$.

3. Cesàro means of $\{n(b_n \cos nx - a_n \sin nx)\}$. We consider the allied Fourier series (2.2) of $f(t)$, and write for $\alpha > -1$

(3.1) $$\bar{\tau}_n^{\alpha+1}(x) = \frac{1}{A_n^{\alpha+1}} \sum_{\nu=1}^n A_{n-\nu}^\alpha \nu (b_\nu \cos \nu x - a_\nu \sin \nu x).$$

Then, the right hand side is written as

$$-\frac{1}{A_n^{\alpha+1}} \frac{2}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} D_n^\alpha(t)\, dt,$$

where $\psi(t) = \psi_x(t)$ is defined in §2. So

(3.2) $$\bar{\tau}_n^{\alpha+1}(x) = -\frac{2}{\pi} \int_0^\pi \psi(t) \left(\frac{\alpha+1}{\alpha+1+n} \frac{d}{dt} \mathrm{K}_n^\alpha(t)\right) dt.$$

Using (3.2) replaced α by $\alpha+k$, and n by $n-k$ we have

(3.3) $$\begin{aligned} &\frac{\alpha(\alpha+1)}{m+1} A_m^{\alpha+1} \sum_{k=0}^m (-1)^k \binom{m}{k} ((\alpha+k)(\alpha+1+k))^{-1} \bar{\tau}_{n-k}^{\alpha+1+k}(x) \\ &= -\frac{\alpha+m+1}{m+1} \frac{2}{\pi} \int_0^\pi \psi(t) \left(\frac{1}{\alpha+1+n} \frac{d}{dt} L_n^\alpha(t)\right) dt, \end{aligned}$$

where $L_n^\alpha(t)=L_n^\alpha(m,t)$ is defined by (2.6), and $(\alpha+1+n)^{-1}(d/dt)L_n^\alpha(t)$ satisfies the conditions (2.8), (2.9) and (2.10) (cf. the proof of Theorem 2). Moreover, we see by an elaboration

$$-\frac{\alpha+m+1}{m+1}\int_0^\pi\Big(\frac{1}{\alpha+1+n}\frac{d}{dt}L_n^\alpha(t)\Big)dt=1+o(1) \qquad (n\to\infty).$$

From (3.2), (3.3) and the preceding Lemma replaced α by $\alpha+1$, and σ's by $\bar{\tau}$'s, we have the following

THEOREM 4. If $\alpha>-1$, then a necessary and sufficient condition for

$$\bar{\tau}_n^{\alpha+1}(x)=s+o(n^{-\delta}) \qquad (n\to\infty),$$

where $\bar{\tau}_n^{\alpha+1}(x)$ is defined by (3.1), i.e. (3.2), and $\delta<2$, is that

$$-\frac{\alpha+m+1}{m+1}\frac{2}{\pi}\int_0^\pi\psi(t)\Big(\frac{1}{\alpha+1+n}\frac{d}{dt}L_n^\alpha(t)\Big)dt=s+o(n^{-\delta}) \qquad (n\to\infty),$$

where $L_n^\alpha(t)=L_n^\alpha(m,t)$ is defined by (2.6), and $(\alpha+1+n)^{-1}(d/dt)L_n^\alpha(t)$ possesses the properties of $L_n^\alpha(t)$, i.e. (2.8), (2.9) and (2.10).

Remark 3. From (3.2) we have readily the following identity:

$$\bar{\tau}_n^{\alpha+1}(x)=\frac{2}{\pi}l+o(1)-\frac{2}{\pi}\int_0^\pi(\psi(t)-l)\Big(\frac{\alpha+1}{\alpha+1+n}\frac{d}{dt}\mathrm{K}_n^\alpha(t)\Big)dt,$$

where $l=l(x)$. Hence roughly speaking, by Theorems 2, 4, and Remark 2, the proposition

$$\bar{\tau}_n^{\alpha+1}(x)\to 2l/\pi, \quad \alpha>-1, \qquad (n\to\infty),$$

may be argued quite analogously as $\sigma_n^\alpha(x)\to s$ does.

References

[1] K. Yano: On summability (C,α), $\alpha\geqq 0$, of Fourier series and allied Fourier series (in Japanese), Jour. Nara Women's Univ., **1**, 1–24 (1951).

[2] K. Yano: On the jump functions, Kôdai Math. Semi. Rep., **9**, 1–11 (1957).

[3] K. Yano: On a method of Cesàro summation for Fourier series, Kôdai Math. Semi. Rep., **9**, 49–58 (1957).

[4] J. J. Gergen: Convergence and summability criteria for Fourier series, Quart. Jour. Math., **1**, 252–275 (1930).

[5] A. Zygmund: Trigonometrical Series, Warszawa-Lwow (1935).