26. On Semi-continuity of Functionals. II

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(Comm. by K. KUNUGI, M.J.A., March 12, 1959)

1. Introduction. In earlier paper [2], we have proved Theorem 1 [2] which is concerned with the semi-continuity of additive functionals on semi-ordered linear spaces. By the same notion, we shall obtain some results concerning additive functionals on Boolean algebras.[1)] Let B be a σ-complete[2)] Boolean algebra. A positive functional m on B is called a *finitely additive measure* if the following condition is satisfied.

(1.1) $$m(x+y)=m(x)+m(y) \quad \text{for } x, y \in B \quad \text{with } x\frown y=0.$$

Furthermore if the functional m satisfies the following condition (1.2), m is called a *totally additive measure.*

(1.2) *For a system of mutually orthogonal elements* x_i $(i=1, 2,\cdots)$ *we have*

$$m(\bigcup_{i=1}^{\infty} x_i) = \sum_{i=1}^{\infty} m(x_i)$$

(1.2) implies (1.1), but the converse does not follow. However, sometimes a finitely additive measure is totally additive on some ideal[3)] of B.

If B is a Boolean algebra, then we can consider the representation space. (This space consists of all dual maximal ideals $\mathfrak{p}$ of B.) We denote this space by $\mathfrak{E}$. $\mathfrak{E}$ constitutes a compact Hausdorff space with open basis: $U_x=\{\mathfrak{p}:\mathfrak{p}\ni x\}$, $x\in B$.

If B is σ-complete, then the closure of a σ-open set (countable union of closed sets) of $\mathfrak{E}$ is open in $\mathfrak{E}$. An ideal I of B is said to be *dense* in B if for any $x(\neq 0)\in B$ there exists an element $y\in I$ with $0\neq y\leqq x$.

We shall consider the following property of σ-complete Boolean algebra.

(A) *Let* A_n $(n=1, 2,\cdots)\subset\mathfrak{E}$ *be* σ*-open and dense. Then we can find an open dense set* $U\subset\mathfrak{E}$ *with* $U\subset\bigcap_{n=1}^{\infty} A_n$.

We have also the following property equivalent to (A).

(A′) *Let* B_n $(n=1, 2,\cdots)\subset\mathfrak{E}$ *be* δ*-closed*[4)] *and no-where dense*

1) For the definition of Boolean algebra, see [1, Chapter 10].

2) B is σ-complete if for x_i $(i=1, 2,\cdots)$, there exists $x=\bigcup_{i=1}^{\infty} x_i$.

3) $M\subset B$ is an ideal (in Birkhoff's terminology [1]) if $a\in M$, $b\leqq a$ implies $b\in M$.

4) Complement of σ-open set.

sets in $\mathfrak{E}$. *Then* $\bigcup_{n=1}^{\infty} B_n$ *is no-where dense.*

2. *Theorem* 1. *Let* B *be a* σ*-complete Boolean algebra with the property* (A) *and let* m *be a finitely additive measure on* B. *Then* m *is totally additive on some dense ideal of* B.

Proof. By the same method applied to the proof of Theorem 1 [2], we can find a σ-open and dense set $A_k \subset \mathfrak{E}$ $(k=1, 2, \cdots)$ such that $A_k \supset U_{x_i}$ $(i=1, 2, \cdots)$ and $x_i \downarrow_{i=1}^{\infty} 0$ implies

$$\inf_i m(x_i) \leqq \frac{1}{k}.$$

By the property (A), we find an open dense set $U \subset \mathfrak{E}$ with $U \subset \bigcap_{k=1}^{\infty} A_k$.

If $U \supset U_{x_i}$ $(x_i=1, 2, \cdots)$, and $x_i \downarrow_{i=1}^{\infty} 0$, we see that $\inf_i m(x_i) \leqq \frac{1}{k}$ $(k=1, 2, \cdots)$, i.e. $\inf_i m(x_i)=0$.

Since U is open and dense in $\mathfrak{E}$, the set $I=\{x : x \in B \text{ and } U_x \subset U\}$ is a dense ideal of B. For mutually orthogonal elements $x_i \in U$ with $\bigcup_{i=1}^{\infty} x_i = x \in U$ and $y_j = \bigcup_{i=j}^{\infty} x_i$ $(i=1, 2, \cdots)$, we have $\bigcap_{j=1}^{\infty} y_j = \bigcap_{j=1}^{\infty} (\bigcup_{i=j}^{\infty} x_i)=0$ and $y_1 \geqq y_2 \geqq \cdots$, hence $\inf_j m(y_j)=0$, i.e. $m(\bigcup_{i=1}^{\infty} x_i)=\sum_{i=1}^{\infty} m(x_i)$. This proves the theorem.

We shall consider another property of a σ-complete Boolean algebra B.

(B) *Let* A_n $(n=1, 2, \cdots)$ *be* σ*-open and dense sets in* $\mathfrak{E}$. *Then* $\bigcap_{n=1}^{\infty} A_n$ *contains a* σ*-open dense set.*

H. Nakano has proved that (B) is equivalent to the following.[5]

(B′) *For double system* $x_{i,j}$ *with* $x_{i,j} \uparrow_j x$; *there exist* x_k $(k=1, 2, \cdots)$ *and number* $n(i, k)$, $i, k=1, 2, \cdots$ *with* $x_k \uparrow_k x$ *and* $x_k \leqq x_{i, n(i,k)}$.

(B) implies (A), but (A) does not follow (B).

(C) 1*st category set in* $\mathfrak{E}$ *is always no-where dense.*

It is easy to see that (C) implies (A).

Remark 1. If B is complete,[6] under the hypothesis of continuum, (B) implies (C).[7]

Corollary 1. *Let* B *have the property* (B) *or* (C) *and* m_n $(n=1, 2, \cdots)$ *be finitely additive measures. Then there exists a dense ideal in which* m_n *are totally additive at the same time.*

Proof. We shall prove only the case that B has the property (B). By Theorem 1 and the property (B), there exist σ-open and dense sets $U_n \subset \mathfrak{E}$ $(n=1, 2, \cdots)$ such that

5) See [3, p. 45].

6) B is complete if for x_λ $(\lambda \in \Lambda) \in B$, there exists $x = \bigcup_{\lambda \in \Lambda} x_\lambda$.

7) This fact is due to Prof. I. Amemiya.

$$U_n \supset U_{x_i}, \ \bigcap_{i=1}^{\infty} x_i = 0, \ x_1 \geqq x_2 \geqq \cdots \text{ imply } \inf_i m_n(x_i) = 0.$$

By the property (B), we find a σ-open and dense set U with $U \subset \bigcap_{n=1}^{\infty} U_n$. Putting $I = \{x : U_x \subset U\}$, I is a dense ideal of B in which m_n $(n=1, 2, \cdots)$ are totally additive at the same time.

Corollary 2. *If B is a complete Boolean algebra with the property* (C), *then, for a finitely additive measure m, there exist a normal measure*[8] *m' and dense ideal I of B such that*

$$m(x) \geqq m'(x) \quad \textit{for } x \in B \textit{ and } \quad m(x) = m'(x) \quad \textit{for } x \in I.$$

Proof. By the method applied to Theorem 1 [2], we can find a dense ideal $I \subset B$ such that for any system x_λ $(\lambda \in \Lambda) \in I$ with $x_\lambda \downarrow_\lambda 0$ we have $\inf_{\lambda \in \Lambda} m(x_\lambda) = 0$.

Since I is a dense ideal of B, for any $x \in B$, there exists a system $x_\lambda \in I$ with $x_\lambda \uparrow_{\lambda \in \Lambda} x$. If there exist $y_\gamma (\gamma \in \Gamma) \in I$ and $x_\lambda (\lambda \in \Lambda) \in I$ with $y_\gamma \uparrow_{\gamma \in \Gamma} x$, $x_\lambda \uparrow_{\lambda \in \Lambda} x$, then

$$\sup_{\lambda \in \Lambda} m(x_\lambda) = \sup_{\gamma \in \Gamma} m(y_\gamma).^{9)}$$

Hence, if we put

$$m'(x) = \sup_{\lambda \in \Lambda} m(x_\lambda) \ \text{ for } \ x = \bigcup_{\lambda \in \Lambda} x_\lambda (x_\lambda \in I),$$

then m' satisfies the conditions of Corollary 2.

Remark 2. Theorem 1 is not true in the case that B has not the property (A). For example, let (0, 1) be an open interval of real numbers with terminals 0, 1. The complete Boolean algebra C consisting of regularly open sets[10] in (0, 1) has not the property (A). For, $\mathfrak{E}$ (the representation space of C) has a dense and countable set $\{\mathfrak{p}_i\}$ $(i=1, 2, \cdots)$ and any element of $\mathfrak{E}$ is not isolated; therefore $\mathfrak{E} - \mathfrak{p}_i = A_i$ is dense in $\mathfrak{E}$, and $\bigcap_{i=1}^{\infty} A_i$ does not contain any open and dense set. Furthermore A_i is σ-open set, i.e. C has not the property (A). Let m be totally additive measure on B. Then m is always 0, i.e. $m(x) = 0$ for every $x \in C$. For any $\mathfrak{p}_i$ $(i=1, 2, \cdots)$, we can find a sequence $x_{ij} \downarrow_j$ $(j=1, 2, \cdots)$ such that

$$\mathfrak{E} \supset U_{x_{ij}} \ni \mathfrak{p}_i \quad \text{and} \quad \bigcap_{j=1}^{\infty} x_{i,j} = 0.$$

If m is totally additive, then we can find j_i with $m(x_{i,j_i}) \leqq \varepsilon \frac{1}{2^i}$ $(i=1, 2, \cdots)$ where ε is an arbitrary given positive number. Because $\{\mathfrak{p}_i\}$ $(i=1, 2, \cdots)$ is dense in $\mathfrak{E}$, we see that

$$1 = \bigcup_{i=1}^{\infty} x_{i,j_i} \quad (1 \text{ is maximal element of } C)$$

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8) m is called a normal measure if $x_\lambda \uparrow_{\lambda \in \Lambda} x$ implies $m(x) = \sup_{\lambda \in \Lambda} m(x_\lambda)$.

9) This fact is independent from the cardinal number of Λ or Γ.

10) E is called a regularly open set if interior of $\overline{E}$ is E.

and

$$m(1)\leqq\sum_{i=1}^{\infty}m(x_{i,j_i})\leqq\varepsilon\Big(\frac{1}{2}+\frac{1}{2^2}+\cdots\Big)=\varepsilon.$$

Since we can choose ε arbitrary, we have $m(1)=0$. Hence $m(x)=0$ for $x\in C$. Furthermore there exists a finitely additive measure m on B such that $m(x)>0$ for $x(\neq 0)\in B$. For instance, putting $f_x(\mathfrak{p}_i)=\frac{1}{2^i}$ if $x\in\mathfrak{p}_i$ and $f_x(\mathfrak{p}_i)=0$ if $x\bar{\in}\mathfrak{p}_i$, we see that $m(x)=\sum_{i=1}^{\infty}f_x(\mathfrak{p}_i)$, $x\in C$ is finitely additive measure on C. Thus C is an example which does not follow Theorem 1.

3. Applications. Let R be a totally continuous and super-universally continuous semi-ordered linear space. H. Nakano studied modulared linear spaces. We shall apply Theorem 1 [2] to finite modulars without proof.

Theorem 2. *Let m be a functional on R which satisfies modular*[11] *conditions except semi-continuity axiom, but is coefficient-continuous.*[12] *Then there exists a complete semi-normal manifold of R in which m satisfies modular conditions.*

R is called semi-regular if $\bar{a}(a)=0$ (for all $\bar{a}\in\bar{R}$)[13] imply $a=0$. In the case that R is semi-regular, we can define $\overline{\overline{m}}$ such that

$$\overline{\overline{m}}(a)=\sup_{\bar{a}\in\bar{R}^m}\{\bar{a}(a)-\overline{m}(\bar{a})\}$$

where $\bar{R}^m$[14] is the modular conjugate space of R and $\overline{m}(\bar{a})=\sup_{a\in R}\{\bar{a}(a)-m(a)\}$ for $\bar{a}\in\bar{R}^m$. Furthermore, if m is a modular, then $\overline{\overline{m}}(a)=m(a)$ for all $a\in R$.

Theorem 3. *Let m be a functional on semi-regular space R which satisfies the conditions of Theorem* 2. *Then $m=\overline{\overline{m}}$ on some complete semi-normal manifold.*

A norm $\|a\|$, $a\in R$ is said to be L-type norm if $a\geqq 0$, $b\geqq 0$ imply $\|a+b\|=\|a\|+\|b\|$. A norm $\|a\|$ is said to be continuous if $a_\lambda\downarrow_{\lambda\in\Lambda}0$ implies $\inf_{\lambda\in\Lambda}\|a_\lambda\|=0$. It is well known that if $\|\ \|$ is complete, then $\|\ \|$ is continuous.

Theorem 4. *If there exists an L-type norm on R, then this norm coincides with some continuous norm on some complete semi-normal manifold.*

Remark 3. Theorems 2, 3, 4 do not remain true if R is not totally continuous. For instance the totality of continuous functions defined

11) For the definition of modulars, see [3].
12) See [2].
13) $\bar{R}$ is the totality of universally continuous linear functionals on R (see [3]).
14) See [3].

on $\mathfrak{E}$ in the former remark is not totally continuous and an example which does not follow Theorems 2, 3, 4.

References

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[3] H. Nakano: Modulared Semi-ordered Linear Space, Tokyo Math., Book series (1950).