33. On the Extensions of Finite Factors. I

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In the previous paper [3], we have introduced the concept of crossed product of von Neumann algebras and using it we have generalized the construction of finite factors given by F. J. Murray and J. von Neumann [2]. However the definition given there is rather restricted comparing with the usual concept given for Notherian rings since it corresponds only for cases with special factor sets. In this note we modify the definition of crossed product of von Neumann algebras to fill up this gap and show some properties of this generalized product.

We shall show in the below that the crossed product of a finite factor with respect to a *normalized* factor set of unitary operators can be defined as usual and that the product is related deeply with the extension of a group described in [1]. Also we shall show that the product satisfies the usual elementary properties of the crossed product and that our new product gives a way of construction of the crossed product basing on an intermediate subfactor.

1. Let $\widetilde{\mathfrak{A}}$ be a discrete group and K be a normal subgroup of $\widetilde{\mathfrak{A}}$. Put $\mathfrak{A}$ the quotient group $\widetilde{\mathfrak{A}}/K$. Now we assume that a group G is isomorphic to a subgroup of $\mathfrak{A}$. (As there is no fear for confusion, we identify G with the subgroup in the following.) For each $\alpha\in G$, we define a selection $\overline{\alpha}\in\alpha$. Throughout the note, the selection $\alpha\to\overline{\alpha}$ will be fixed unless the contrary is stated explicitly. Since K is normal, $k^{\alpha}=\overline{\alpha}^{-1}k\overline{\alpha}\in K$ for every $k\in K$ and for every $\alpha\in G$, and so α determines an automorphism of the group K. (Every element m of K induces an inner automorphism of K such that $k^{m}=m^{-1}km$.) By the choice of $\overline{\alpha}$ and $\overline{\beta}$, there is an element $m_{\alpha,\beta}\in K$ satisfying the relation

$$\overline{\alpha}\cdot\overline{\beta}=\overline{\alpha\beta}\cdot m_{\alpha,\beta}.$$

Considering α and β as automorphisms of K, the above relation gives

$$(1)\qquad (k^{\alpha})^{\beta}=(k^{\alpha\beta})^{m_{\alpha,\beta}} \qquad \text{for } k\in K.$$

From the definition of the automorphism γ of K, $m_{\alpha,\beta}\overline{\gamma}=\overline{\gamma}m^{\gamma}_{\alpha,\beta}$. By this equality and the associativity of group operation for $\widetilde{\mathfrak{A}}$, the family $\{m_{\alpha,\beta}\mid\alpha,\beta\in G\}$ of elements in K satisfies the following equality:

$$(2)\qquad m_{\alpha,\beta\gamma}m_{\beta,\gamma}=m_{\alpha\beta,\gamma}m^{\gamma}_{\alpha,\beta}$$

for every α, β and γ in G. Such a family $\{m_{\alpha,\beta}\mid\alpha,\beta\in G\}$ satisfying (2) is named *a factor set of* K. By a few simple calculations, we have $k^1=k^{m_{1,1}}$, $m_{\alpha,1}=m_{1,1}$ and $m_{1,\gamma}=m_{1,1}^{\gamma}$. If $m_{1,1}=1$ is satisfied, then we shall call it *weakly normalized*. If moreover $m_{\alpha,\alpha^{-1}}=1$ is satisfied for each $\alpha\in G$, the factor set is called *normalized*. We are mainly concerned with normalized factor sets in this paper.

2. As seen in [1],[1] associating with a group G and a normalized factor set $\{m_{\alpha,\beta}\}$, an extension $\boldsymbol{K}=(K, G, m_{\alpha,\beta})$ of the group K by G is given as follows: Let $\boldsymbol{K}$ be the collection of all symbols $\alpha\otimes k$, where $\alpha\in G$ and $k\in K$. We define multiplication in $\boldsymbol{K}$ by

$$(\alpha\otimes k)(\beta\otimes h)=\alpha\beta\otimes m_{\alpha,\beta}k^{\beta}h.$$

Then the associative law of multiplication follows from its definition and conditions (1) and (2). $1\otimes 1$ is the unit element of $\boldsymbol{K}$, and $\alpha^{-1}\otimes(k^{\alpha^{-1}})^{-1}$ is the inverse of $\alpha\otimes k$, that is, $\boldsymbol{K}$ is a group with the multiplication. The elements of the form $1\otimes k$ in $\boldsymbol{K}$ form a subgroup which is isomorphic to K. We identify this subgroup with K in the following. We get

$$(\alpha\otimes 1)(1\otimes k)=(\alpha\otimes k),$$
$$(\alpha\otimes 1)^{-1}(1\otimes k)(\alpha\otimes 1)=(1\otimes k^{\alpha}).$$

Thus K is a normal subgroup of $\boldsymbol{K}$ and $\boldsymbol{K}/K\cong G$.

Two extensions $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ of K by G are called *equivalent* if there exists an isomorphism between $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ such that it coincides with the identity automorphism on K embedded in $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ and it maps the cosets of K in $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ corresponding to the same $\alpha\in G$ onto each other.

In two extensions $\boldsymbol{K}_1=(K, G, m^{(1)}_{\alpha,\beta})$ and $\boldsymbol{K}_2=(K, G, m^{(2)}_{\alpha,\beta})$, if for every $\alpha\in G$, an element c_{α} of K is associated in such a way that $\overline{\alpha}_{(2)}=\overline{\alpha}_{(1)}c_{\alpha}$ for selections $\overline{\alpha}_{(1)}, \overline{\alpha}_{(2)}$ of each cases, then factor sets $\{m^{(1)}_{\alpha,\beta}\}$ and $\{m^{(2)}_{\alpha,\beta}\}$ are linked by the relations

$$m^{(2)}_{\alpha,\beta}=c_{\alpha\beta}^{-1}m^{(1)}_{\alpha,\beta}c_{\alpha}^{\beta}c_{\beta}. \tag{3}$$

These two factor sets linked by the relation (3) are said *equivalent*. It is known that $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are equivalent if and only if $\{m^{(1)}_{\alpha,\beta}\}$ and $\{m^{(2)}_{\alpha,\beta}\}$ are equivalent. Every factor set $\{m_{\alpha,\beta}\}$ is equivalent to a weakly normalized factor set. If G has no element of oder 2, every factor set is equivalent to a normalized one, because we may choose $\overline{\alpha}, \overline{\alpha}^{-1}$ such as $\overline{\alpha}\cdot\overline{\alpha}^{-1}=1$.

3. Hereafter denote by A a finite factor on a separable Hilbert space and take $\widetilde{\mathfrak{A}}$ the group of all automorphisms of A. (An automorphism in this paper means always a $*$-automorphism.) The inner automorphisms become a normal subgroup K in $\widetilde{\mathfrak{A}}$. We put $\mathfrak{A}=\widetilde{\mathfrak{A}}/K$ and G an *enumerable* subgroup of $\mathfrak{A}$, then the discussions mentioned

1) A general theory of group extensions is stated in Kurosh's book [1]. We need here special extensions undergoing some additional conditions.

above are all applicable for the present $\tilde{\mathfrak{A}}$, $\mathfrak{A}$ and G.

An inner automorphism m of A is realized by a unitary operator $v \in A$, that is, there exists v such that $a^m = v^* a v$. But this v is not determined uniquely. A unitary operator w satisfying $w^* v = \chi \cdot 1$ (χ is a complex number such that $|\chi| = 1$) induces the same inner automorphism m. If $v_{\alpha,\beta}$ is a unitary operator which induces the inner automorphism $m_{\alpha,\beta}(\alpha, \beta \in G)$ then, by the relation (2), there exists a complex number $\chi(\alpha, \beta, \gamma)$ such that

$$v_{\alpha\beta,\gamma} v_{\alpha,\beta} \cdot \chi(\alpha, \beta, \gamma) = v_{\alpha\beta,\gamma} v^{\gamma}_{\alpha,\beta}, \quad |\chi(\alpha, \beta, \gamma)| = 1$$

where $v^{\gamma}_{\alpha,\beta}$ shows the image of $v_{\alpha,\beta} \in A$ due to the automorphism $\bar{\gamma}$. When we can take $\chi(\alpha, \beta, \gamma) = 1$, the set of unitary operators $\{v_{\alpha,\beta}\}$ satisfies

$$(4) \qquad (x^{\alpha})^{\beta} = v^*_{\alpha,\beta} x^{\alpha\beta} v_{\alpha,\beta}$$

and

$$(5) \qquad v_{\alpha,\beta\gamma} v_{\beta,\gamma} = v_{\alpha\beta,\gamma} v^{\gamma}_{\alpha,\beta}.$$

Such a system $\{v_{\alpha,\beta}\}$ is called a *factor set of unitary operators of* A. When a factor set $\{v_{\alpha,\beta}\}$ satisfies further

$$(6) \qquad v_{1,1} = 1, \quad v_{\alpha,\alpha^{-1}} = v_{\alpha^{-1},\alpha} = \lambda_{\alpha} \cdot 1$$

(λ_{α} is a complex number such that $|\lambda_{\alpha}| = 1$), we call it *normalized* one.

In this section we show that *if for a normalized factor set* $\{m_{\alpha,\beta}\}$ *of inner automorphisms of* A, *a normalized factor set* $\{v_{\alpha,\beta}\}$ *of unitary operators in* A *is associated, an extension algebra of* A *is defined, on which the extended group* $\boldsymbol{K} = (K, G, m_{\alpha,\beta})$ *is faithfully represented as an inner automorphism group.*

As in [3], we denote a function on G with value a_{α} in A at α by $\sum_{\alpha} \alpha \otimes a_{\alpha}$. Let D be the set of all functions $\sum_{\alpha} \alpha \otimes a_{\alpha}$ such that $a_{\alpha} = 0$ except a finite number of α's. Then D becomes a $*$-algebra by

multiplication: $\quad (\sum_{\alpha} \alpha \otimes a_{\alpha})(\sum_{\beta} \beta \otimes b_{\beta}) = \sum_{\alpha,\beta} \alpha\beta \otimes v_{\alpha,\beta} a^{\beta}_{\alpha} b_{\beta}$,

$*$-operation: $\quad (\sum_{\alpha} \alpha \otimes a)^* = \sum_{\alpha} \alpha^{-1} \otimes a^{\alpha^{-1}*}_{\alpha} v^*_{\alpha,\alpha^{-1}}$

According to (5), the multiplication is associative and $1 \otimes 1$ [2)] is the identity of D. By the correspondence $a \leftrightarrow 1 \otimes a$, A is isomorphic to the subalgebra composed of elements $\{1 \otimes a\}$ in D and so we identify A with this subalgebra. Since

$$(\alpha \otimes 1)^* (1 \otimes a)(\alpha \otimes 1) = (1 \otimes a^{\alpha}),$$
$$(1 \otimes u_k)^* (1 \otimes a)(1 \otimes u_k) = (1 \otimes a^k)$$

where u_k is a unitary operator in A such that $a^k = u^*_k a u_k$, and since

$$(\alpha \otimes 1)(1 \otimes u_k) = \alpha \otimes u_k,$$

we know D is a $*$-algebra admitting an inner automorphism group isomorphic to $\boldsymbol{K} = (K, G, m_{\alpha,\beta})$ under which the subalgebra A remains invariant.

2) Simply by $\alpha \otimes a$ we show the function in D such that $a_{\alpha} = a$ and $a_{\beta} = 0$ for $\beta \neq \alpha$.

Let τ be the trace of A, we put

$$\bar{\tau}(\alpha\otimes a_\alpha)=\begin{cases}\tau(a_\alpha) & \text{if } \alpha=1,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\bar{\tau}(\sum_\alpha \alpha\otimes a_\alpha)=\sum_\alpha \bar{\tau}(\alpha\otimes a_\alpha).$$

Then $\bar{\tau}$ is a faithful trace on D, and as similar as in [3], we get a von Neumann algebra $\boldsymbol{A}$ as the weak or metric closure of D due to the trace $\bar{\tau}$, which we call *the crossed product of A by G with respect to a factor system $\{v_{\alpha,\beta}\}$ of unitary operators* and denote by $(A, G, v_{\alpha,\beta})$. When G is represented faithfully as an outer automorphism group of A, we can take the selection $\bar{\alpha}$ in such a way that they form a group isomorphic to G. Then $m_{\alpha,\beta}=1$ for every α and β. Undoubtedly this is a normalized factor set of inner automorphisms of A and as an associated normalized factor set of unitary operators, we may take $v_{\alpha,\beta}=1$ for every α and β. The crossed product of A by G with respect to this factor set $\{v_{\alpha,\beta}\}$ is nothing but the crossed product $G\otimes A$ discussed in [3].

4. Now we shall show an extension of [3, Theorem 1]:

THEOREM 1. *The crossed product $\boldsymbol{A}=(A, G, v_{\alpha,\beta})$ of a finite factor A by an enumerable group G with respect to a normalized factor set $\{v_{\alpha,\beta}\}$ is a finite factor, which admits an inner automorphism group isomorphic to the extension $\boldsymbol{K}=(K, G, m_{\alpha,\beta})$ of the group K of all inner automorphisms of A.*

Proof. Put $e=\sum_\alpha \alpha\otimes a_\alpha$ a non-trivial central projection of $\boldsymbol{A}$. For $x\in A$, $ex=xe$ means

$$v_{\alpha,1}a_\alpha^1 x=v_{1,\alpha}x^\alpha a_\alpha.$$

This gives $a_\alpha x=x^\alpha a_\alpha$. By [3, Lemma 1], $a_\alpha=0$ unless $\alpha=1$. This contradicts the assumption about e and so $\boldsymbol{A}$ is a factor of finite type since it has a faithful trace. By the construction, unitary operators $\{\alpha\otimes u_k\}$ give a group of inner automorphisms isomorphic to $\boldsymbol{K}$. q.e.d.

Hereafter, we identify $\boldsymbol{K}$ with the above-mentioned inner automorphism group of $\boldsymbol{A}$.

5. Here we shall show an extension of [3, Theorem 2]:

THEOREM 2. *If a von Neumann subalgebra $\boldsymbol{B}$ of the crossed product $\boldsymbol{A}=(A, G, v_{\alpha,\beta})$ of a finite factor A by an enumerable group G with respect to $\{v_{\alpha,\beta}\}$ contains A, then $\boldsymbol{B}$ is the crossed product $(A, F, v_{\alpha,\beta})$ of A by a subgroup F of G, in the latter, $v_{\alpha,\beta}$ are restricted for elements α, β in F.*

Proof. Let $u_\beta^\dagger$ be the expectation of $u_\beta=\beta\otimes 1\in D$ conditioned by $\boldsymbol{B}$, and $u_\beta^\dagger=\sum_\alpha \alpha\otimes c_\alpha$ be the Fourier expansion of $u_\beta^\dagger$. Since $x\in A$ is in $\boldsymbol{B}$, $xu_\beta^\dagger=u_\beta^\dagger x^\beta$. Then Fourier expansion of $xu_\beta^\dagger$ is $\sum_\alpha \alpha\otimes x^\alpha c_\alpha$ and that of $u_\beta^{\dagger}x^\beta$ is $\sum_\alpha \alpha\otimes c_\alpha x^\beta$. Thus $x^\alpha c_\alpha = c_\alpha x^\beta$ for all α. This means

$$c_\alpha=\begin{cases} 0 & \text{if } \alpha\neq\beta, \\ \lambda\cdot 1 & \text{if } \alpha=\beta \end{cases}$$

(λ is a complex number). Therefore $u_\beta^{\dagger}=\beta\otimes\lambda\cdot 1$, and so either $u_\beta^{\dagger}=0$ or $u_\beta=\beta\otimes 1\in \boldsymbol{B}$.

Let F be the collection of β such that $u_\beta\in\boldsymbol{B}$, then F is a subgroup of G. Now we can construct the crossed product $(A, F, v_{\alpha,\beta})_{\alpha,\beta\in F}$, which is a subalgebra of $\boldsymbol{B}$. If $\boldsymbol{B}$ does not coincide with $(A, F, v_{\alpha,\beta})$, there is an element $b=\sum_\alpha \alpha\otimes b_\alpha$ such that $b_\gamma\neq 0$ for at least one γ excluded from F. Now take up $c=1\otimes b_\gamma^*\in A$.

$$b\cdot c=\sum_\alpha \alpha\otimes v_{\alpha,1}b_\gamma b_\gamma^*=\sum_\alpha \alpha\otimes b_\gamma b_\gamma^*,\quad \bar{\tau}(b\cdot c\cdot u_{\gamma^{-1}})=\lambda_\gamma\cdot\tau(b_\gamma b_\gamma^*).$$

On the other hand, since $\gamma^{-1}\notin F$, $u_{\gamma^{-1}}^{\dagger}=0$, that is, γ^{-1} is orthogonal to $\boldsymbol{B}$. This shows $\bar{\tau}(b\cdot c\cdot u_{\gamma^{-1}})=0$ and $b_\gamma=0$. This is a contradiction. Hence $\boldsymbol{B}=(A, F, v_{\alpha,\beta})$.

COROLLARY. *A subfactor $\boldsymbol{B}=(A, F, v_{\alpha,\beta})$ of the crossed product $\boldsymbol{A}=(A, G, v_{\alpha,\beta})$, which contains A, is invariant under the inner automorphisms belonging to $\boldsymbol{K}=(K, G, m_{\alpha,\beta})$ if and only if the corresponding subgroup F is normal in G.*

Proof. Let $b=\sum_\alpha \alpha\otimes b_\alpha$ be an element of $\boldsymbol{B}$. For $u_\alpha=\alpha\otimes 1$, $u_\alpha^*=\alpha^{-1}\otimes v_{\alpha,\alpha^{-1}}^*$ and

$$u_\alpha^* b u_\alpha=\sum_\alpha \beta^{-1}\alpha\beta\otimes v_{\beta^{-1},\alpha\beta}(v_{\beta,\beta^{-1}}^*)^{\alpha\beta}v_{\alpha,\beta}b_\alpha^\beta.$$

By Theorem 2, $u_\alpha^* b u_\alpha\in\boldsymbol{B}$ if and only if $\beta^{-1}\alpha\beta\in F$. q.e.d.

6. Let $\boldsymbol{A}=(A, G, 1)$ be a crossed product of a finite factor A by a group G with respect to a factor set satisfying $v_{\alpha,\beta}=1$ for every pair of $\alpha,\beta\in G$. This is the crossed product $G\otimes A$ discussed in [3]. We assume that $\boldsymbol{B}=H\otimes A$ is a subfactor of $\boldsymbol{A}$ corresponding to a normal subgroup H of G. Then the group is represented as a group of unitary operators by the correspondence $h\in H\leftrightarrow h\otimes 1\in\boldsymbol{B}$. Now denote by Θ the quotient group G/H and by $\theta,\pi,\rho,\cdots$ elements of Θ. For each θ, we make a selection α_θ from the corresponding coset of H.[3] Here we assume that $\alpha_1=1$ and $\alpha_\theta\cdot\alpha_{\theta^{-1}}=1$ for every θ. For each pair of $\theta,\pi\in\Theta$ there is an element $h_{\theta,\pi}$ in H such that $\alpha_\theta\cdot\alpha_\pi=\alpha_{\theta\pi}\cdot h_{\theta,\pi}$. By the associativity of the group operation for Θ

$$h_{\theta,\pi\rho}h_{\pi,\rho}=h_{\theta\pi,\rho}h_{\theta,\pi}^\rho$$

where $h^\rho=\alpha_\rho^{-1}h\alpha_\rho$. Since $h_{\theta,\pi}\in H$, there is a unitary operator $u_{\theta,\pi}=(h_{\theta,\pi}\otimes 1)$ in $\boldsymbol{B}$ such that

$$u_{\theta,\pi\rho}u_{\pi,\rho}=u_{\theta\pi,\rho}u_{\theta,\pi}^\rho,\quad u_{\theta,\theta^{-1}}=u_{\theta^{-1},\theta}=1$$

for every θ,π,ρ. This shows that a normalized factor set $\{u_{\theta,\pi}\}$ of unitary operators in $\boldsymbol{B}$ is associated to the normalized factor set $\{h_{\theta,\pi}\}$ of inner automorphisms of $\boldsymbol{B}$. Hence we can construct the crossed

3) We notice that every α_θ except $\theta=1$ is an outer automorphism of $\boldsymbol{B}$.

product $(\boldsymbol{B}, \Theta, u_{\theta,\pi})$ of $\boldsymbol{B}$ by Θ with respect to $\{u_{\theta,\pi}\}$.

THEOREM 3. *Let G be an enumerable outer automorphism group of a finite factor A and H be a normal subgroup of G. If for every element θ of the quotient group $\Theta = G/H$, a representative α_θ can be chosen from the coset corresponding to θ satisfying $\alpha_1 = 1$, $\alpha_\theta \cdot \alpha_{\theta^{-1}} = 1$ then the crossed product $\boldsymbol{A} = G \otimes A$ is isomorphic to $(\boldsymbol{B}, \Theta, u_{\theta,\pi})$.*

Proof. Let D_0 be the collection of A-valued function $\sum_h h \otimes a_h$ defined on H such that $a_h = 0$ except finite set of h's and $\mathfrak{D}$ be the collection of D_0-valued function $\sum_\theta \theta \otimes b_\theta$ defined on Θ satisfying the similar condition as D_0. D be the same as in § 3. Then D and $\mathfrak{D}$ are dense $*$-subalgebra in $G \otimes A$ and $(\boldsymbol{B}, \Theta, u_{\theta,\pi})$ respectively. Every element in $\mathfrak{D}$ is expanded as $\sum_\theta \theta \otimes b_\theta = \sum_\theta \theta \otimes (\sum_h h \otimes a_{\theta h}) = \sum_{\theta,h} \theta h \otimes a_{\theta h}$. We correspond to this an element $\sum_\alpha \alpha \otimes a_\alpha = \sum_{\theta,h} \alpha_\theta h \otimes a_{\theta h}$ in D. (We notice that every element α is uniquely expressed as $\alpha_\theta h$.) This correspondence is one-to-one, multiplication and $*$-preserving. In fact, for $\theta \otimes b_\theta$ and $\pi \otimes c_\pi$ in $\mathfrak{D}$, where $b_\theta = h \otimes a_{\theta h}$, $c_\pi = k \otimes a_{\pi k}$,

$$
\begin{aligned}
(\theta \otimes b_\theta)(\pi \otimes c_\pi) = (\theta\pi \otimes u_{\theta,\pi} b_\theta^\pi c_\pi) &= \theta\pi \otimes [(h_{\theta,\pi} \otimes 1)(h \otimes a_{\theta h})^\pi (k \otimes a_{\pi k})] \\
&= \theta\pi \otimes [(h_{\theta,\pi} \otimes 1)(\alpha_\pi^{-1} h \alpha_\pi \otimes a_{\theta h}^{\alpha_\pi})(k \otimes a_{\pi k}) \\
&= \theta\pi \otimes [h_{\theta\pi} \cdot \alpha_\pi^{-1} h \alpha_\pi \cdot k \otimes a_{\theta h}^{\alpha_\pi k} a_{\pi k}].
\end{aligned}
$$

On the other hand

$$
(\alpha_\theta h \otimes a_{\theta h})(\alpha_\pi k \otimes a_{\pi k}) = \alpha_\theta h \alpha_\pi k \otimes a_{\theta h}^{\alpha_\pi k} a_{\pi k}.
$$

This is the element in D corresponding to $(\theta \otimes b_\theta)(\pi \otimes c_\pi)$. Thus multiplication is preserved by this correspondence.

$$
\begin{aligned}
(\theta \otimes b_\theta)^* = \theta^{-1} \otimes b_\theta^{\theta^{-1}*} = \theta^{-1} \otimes (h \otimes a_{\theta h})^{\theta^{-1}*} &= \theta^{-1} \otimes (\alpha_\theta h \alpha_\theta^{-1} \otimes a_{\theta h}^{\alpha_\theta^{-1}})^* \\
&= \theta^{-1} \otimes (\alpha_\theta h^{-1} \alpha_\theta^{-1} \otimes a_{\theta h}^{h^{-1}\alpha_\theta^{-1}*})
\end{aligned}
$$

which corresponds to $h^{-1}\alpha_\theta^{-1} \otimes a_{\theta h}^{h^{-1}\alpha_\theta^{-1}*} = (\alpha_\theta h \otimes a_{\theta h})^*$ in D. Hence $*$-operation is preserved.

By the definition of the traces in D and $\mathfrak{D}$, it is easily seen that the isomorphism between D and $\mathfrak{D}$ is extended to the one between $G \otimes A$ and $(\boldsymbol{B}, \Theta, u_{\theta,\pi})$. q.e.d.

COROLLARY. *In Theorem 3, if G is the direct product of subgroups F and H, then $G \otimes A$ is isomorphic to $F \otimes (H \otimes A)$.*

Proof. In the present case, Θ of Theorem 3 is isomorphic to F, whence it is possible to choose the elements of F as the representatives of the classes of Θ. Thus, $\alpha_\theta \alpha_\pi = \alpha_{\theta\pi}$ and the factor set becomes trivial. q.e.d.

References

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