# 52. On Locally Q-complete Spaces. I

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If $Z$ is a $Q$-space containing $X$ as a dense subspace, then we call $Z$ a *$Q$-completion of $X$.*[1] $X$ is said to be *locally complete with respect to the structure*[2] $\mu$ of $X$ if for any point $x\in X$, there is a neighborhood whose closure is complete with respect to $\mu$. If $\mu$ is the structure generated by $C(X)$,[2] then we say that $X$ is *locally $Q$-complete.* $X$ is called a local $Q$-space if for any point $x\in X$, there is a neighborhood of $x$ whose closure is a $Q$-space. It is obvious that any $Q$-space is locally $Q$-complete and any locally $Q$-complete space is a local $Q$-space. If $X$ is normal and is a local $Q$-space, then $X$ is locally $Q$-complete.

In this paper, we shall establish some relations between a locally $Q$-complete space and its $Q$-completion, which are analogous to the relations between a locally compact space and its compactification.

**Lemma 1.** *Let $B$ be a closed subset of $X$ and $Z$ a space obtained from $X$ by contracting $B$ to a point. If either $X$ is normal*[3] *or $B$ is compact, then $Z$ is completely regular.*

This lemma is easily proved by the normality of $X$ or the compactness of $B$ respectively. In general, the space $Z$ mentioned above is not necessarily completely regular.

**Lemma 2.** *Let $Y$ be a $Q$-space and $F$ a closed subset of $Y$, and $Z$ be a space obtained from $Y$ by contracting $F$ to a point $p$ in $Y$. If $Z$ is completely regular, then $X=Z-\{p\}$ is locally complete with respect to the structure generated by $C_0=\{f; f\in C(Z),\ f(p)=0\}$.*

*Proof.* We notice first that $C_0$ is considered as a subring $C_1$ of $C(Y)$ whose elements vanish at every point of $B=F\cup\{p\}$. For any point $x$ in $X$, there is a neighborhood $V$ such that $\overline{V}(\text{in } Z)\not\ni p$ in $Z$. To prove that $\overline{V}(\text{in } Z)$ is complete with respect to the structure generated by $C_0$, it is sufficient to prove that $U=\overline{V}(\text{in } Z)$, considered as a closed subset of $Y$, is complete with respect to the structure $\mu$

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1) A space considered here is always a completely regular $T_1$-space. $C(X)$ denotes the totality consisting of all real-valued continuous functions defined on $X$, and $B(X)$ denotes a subset of $C(X)$ consisting of all bounded functions.

2) A *structure* of $X$ considered here means a uniformity of $X$ which agrees the given topology of $X$. A *structure generated by $C$*, which is a subset of $C(X)$, is a structure given by the following uniform neighborhoods: $W(x; f_1,\cdots,f_n,\varepsilon)=l\{y; |f_i(x)-f_i(y)|<\varepsilon\}$ where $f_i\in C$ and $\varepsilon$ is an arbitrary positive real number.

3) In case $X$ is normal, $Z$ is normal.

generated by $C_1$. Let $\{a_\alpha; a_\alpha \in U, \alpha \in \Gamma\}$ be a Cauchy directed set with respect to $\mu$, and $f_1, f_2, \cdots, f_n \in C(Y)$, $\varepsilon > 0$. Since $Z$ is completely regular, there exists $f \in C_1$ such that $f(B)=1$, $f(U)=1$ and $0 \leq f \leq 1$. Then we have $f_i f \in C_1$ for every $i$. Since $\{a_\alpha; \alpha \in \Gamma\}$ is a Cauchy directed set, there is a point $q$ in $U$ and an index $\alpha_0$ such that

$$W_1 = W(q; f_1 f, f_2 f, \cdots, f_n f, \varepsilon)$$
$$= \{x; |f_i f(x) - f_i f(q)| < \varepsilon, x \in U\} \ni a_\alpha \text{ for } \alpha > \alpha_0.$$

By the method of the construction of $f$, we have

$$W(q; f_1, f_2, \cdots, f_n, \varepsilon) \ni a_\alpha \text{ for } \alpha > \alpha_0.$$

This means that $\{a_\alpha; \alpha \in \Gamma\}$ is a Cauchy directed set with respect to the structure generated by $C(Y)$. Since $Y$ is a $Q$-space, $\{a_\alpha; \alpha \in \Gamma\}$ must converge to a point in $U$ because $U$ is closed in $Y$. Thus $\bar{V}$ (in $Z$) is complete with respect to the structure generated by $C_0$.

Since $C(\bar{V}(\text{in } Z))$ can be considered as a set containing $C_0$ as a subset, the space $X$ mentioned in Lemma 2 is locally $Q$-complete, and hence is a local $Q$-space.

**Theorem 1.** *Let $Y, F, p, Z, X$ and $C_0$ be the same as in Lemma 2. If either $F$ is compact or $Y$ is normal, then $X$ is locally complete with respect to the structure generated by $C_0$, and hence $X$ is a local $Q$-space. Moreover $Z$ is a $Q$-space.*

*Proof.* The first part of theorem is an immediated consequence of Lemmas 1 and 2, and hence we shall prove the latter part. $C(Z)$ can be considered as a subring of $C(\nu X)$[4] consisting of all functions which take a constant value on $B$ (as in Lemma 2). Let $(a_\alpha; \alpha \in \Gamma)$ be a Cauchy directed set with respect to the structure $\mu$ generated by $C(Z)$. For any $f_1, f_2, \cdots, f_n \in C(Z)$ and any $\varepsilon > 0$, there exist a point $q$ and an index $\alpha_0$ such that

$$U = W(q; f_1, f_2, \cdots, f_n, \varepsilon) \ni a_\alpha \text{ for } \alpha > \alpha_0.$$

If $|f_i(p) - f_i(q)| < \varepsilon/2$ for each $i$, we have $W(p; f_1, f_2, \cdots, f_n, \varepsilon) \ni a_\alpha$ for $\alpha > \alpha_0$. This means that $\{a_\alpha; \alpha \in \Gamma\}$ converges to $p$. Now suppose that there are $\varepsilon$ and some $f_i$ such that $|f_i(p) - f_i(q)| \geq \varepsilon/2$. Then $\bar{U}(\text{in } Z)$ is disjoint from the point $p$, and hence $\bar{U}(\text{in } Z)$ is complete with respect to $\mu$, because $\bar{U}(\text{in } Z)$ is complete with respect to the structure generated by $C_0$ as easily seen in the proof of Lemma 2. Thus $\{a_\alpha; \alpha \in \Gamma\}$ converges to a point in $\bar{U}(\text{in } Z)$. Therefore $Z$ is a $Q$-space.

Let $\mu$ be the structure generated by a subset of $C(X)$ and $\tilde{X}$ be the completion of $X$ with respect to the structure $\mu$; then we can not

4) For a space $X$ there exists a unique space $\nu X$ which is completely determined, up to homeomorphism by the following properties: (1) $\nu X$ is a $Q$-space, (2) $\nu X$ contains $X$ as a dense subspace, and (3) every function in $C(X)$ can be continuously extended over $\nu X$ (E. Hewitt: Rings of real-valued continuous functions I, Trans. Amer. Math. Soc., **64**, 45–99 (1948)).

conclude that $X$ is open in $\tilde{X}$ even if $X$ is a local $Q$-space. Such an illustration is given by the set of all rational numbers with the structure defined by the usual distance function. The completion $\tilde{X}$ of $X$ is the space of all real numbers with the usual metric, and $X$ is not open in $\tilde{X}$ even if $X$ is a $Q$-space (and hence $X$ is local $Q$-space). The openness of $X$ in $\tilde{X}$ will be investigated in Theorems 2, 3 and 4.

**Lemma 3.** *If $\nu X-X$ is closed in $\nu X$, then $Y=(\nu X-X)^{\beta}\cup X$ is a $Q$-space.*

*Proof.* Now suppose that $Y$ is not a $Q$-space and $b\in\nu Y-Y$. $X\subset Y\subset\beta X$ implies that $\nu Y$ is contained in $\beta X$. By the definition of $\nu Y$, any function of $C(Y)$ is continuously extended over $b$. But we shall prove that this is a contradiction, that is, there is a function of $C(Y)$ which is not continuously extended over $b$. Since $\nu X\not\ni b$, there is a function $f$ of $C(X)$ which is not continuously extended over $b$. On the other hand, the compactness of $(\nu X-X)^{\beta}$, which is disjoint from $X$, implies that there exists a (bounded) function $g$ of $C(\beta X)$ such that $g$ vanishes on some neighborhood $U$(in $\beta X$) of $B$ and $g=1$ on some neighborhood $V$(in $\beta X$) of $b$. Then

$$h=\begin{cases} g \text{ on } U\frown Y \\ gf \text{ on } X-U \end{cases}$$

is a continuous function defined on $Y$. But $gf$ is equal to $f$ on $V\frown X$ and it is not bounded on $V\frown X$, and hence $h$ is not continuously extended over $b$. Thus we have $\nu Y=Y$, that is, $Y$ is a $Q$-space.

A one-point $Q$-completion of $X$ which is not a $Q$-space is a $Q$-space $Z$ such that $Z$ contains $X$ as a dense subset and $Z-X$ consists of only one point. A one-point $Q$-completion of a locally $Q$-complete space which is not a $Q$-space is not necessarily unique. Such an illustration is given by a locally compact space $X$ which is not a $Q$-space. A one-point compactification of $X$ is a one-point $Q$-completion of $X$. On the other hand a $Q$-space $Z$ obtained in ($2\to3$) of Theorem 2 is so also. As easily seen from the proof of ($2\to3$) of Theorem 2, if $X$ is a locally compact space which is not a $Q$-space, and $\beta X-((\nu X-X)^{\beta}\cup X)$ is not a finite set, then there exist infinitely many one-point $Q$-completion of $X$. For any $b\in\beta X-((\nu X-X)^{\beta}\cup X)$, $X\cup(\nu X-X)^{\beta}\cup\{b\}$ becomes a $Q$-space by the same method as in the proof of Lemma 3. We replace $Y$, $F$ and $p$ as in Theorem 1 respectively by $X\cup(\nu X-X)^{\beta}\cup\{b\}$, $(\nu X-X)^{\beta}\cup\{b\}$ and $b$ respectively. Then the space $Z$ is a non-point $Q$-completion of $X$. We shall say $Z$ obtained in ($2\to3$) of Theorem 2, a *natural one-point $Q$-completion of $X$.*

**Theorem 2.** *The following conditions for a non $Q$-space $X$ are equivalent:*

1) *$X$ is locally $Q$-complete,*

2) *$X$ is open in $\nu X$,*

3) *there is a one-point $Q$-completion of $X$.*

*Proof* $(1\to 2)$. We suppose that $B=\nu X-X$ is not void and is not closed in $\nu X$. There is a neighborhood $\overline{V}$(in $\nu X$) of a point $p$ in $\overline{B}$(in $\nu X)\frown X$ such that $\overline{V}$(in $X$) is complete with respect to the structure $\mu$ generated by $C(X)$, and $\overline{V}$(in $X$) contains a direct set $\{a_\alpha;\alpha\in\Gamma\}$ which converges to $p$ where $a_\alpha\in U=V\frown X$ for each $\alpha\in\Gamma$. Since $\{a_\alpha;\alpha\in\Gamma\}$ is a Cauchy directed set in $U$ with respect to the structure $\mu$, there is a point $x$ such that $\{a_\alpha;\alpha\in\Gamma\}\to x\in\overline{U}$(in $X$). This is a contradiction, and hence $B$ must be closed in $\nu X$.

$(2\to 3)$. Suppose that $B=\nu X-X$ is closed in $\nu X$. Now we consider $\nu X$ as a subspace of $\beta X$(=the Čech compactification of $X$). Then $\overline{B}$(in $\beta X)=B_0$ is compact and is disjoint from $X$ by a closedness of $B$. We replace $Y$, $F$ and $p$ as in Theorem 1 respectively by $\nu X\smile B_0$, $B_0$ and a point $p$ in $B_0$ respectively. Then $\nu X\smile B_0$ is a $Q$-space by Lemma 3 and the space $Z$ as in Lemma 2 is completely regular. On the other hand, $Z$ can be considered as a continuous image of $\nu X$ under a mapping $\varphi\psi$ where $\psi$ is an identical mapping from $\nu X$ into $\nu X\smile B_0$ and $\varphi$ is a mapping from $\nu X\smile B_0$ onto $Z$ such that $\varphi(x)=x$ for $x\notin B_0$ and $\varphi(x)=p$ for $x\in B_0$. Therefore, by Lemmas 1 and 2, $Z$ is a $Q$-space and $Z=X\smile(p)$, i.e. $Z$ is a (natural) one-point $Q$-completion of $X$.

$(3\to 1)$. Since $X$ has a one-point $Q$-completion $Z$, it is obvious that $X$ is open in the space $Z$. By Lemma 2 $X=Z-\{p\}$ is locally complete with respect to the structure generated by a subset consisting of functions which vanish at the point $p$. This subset is a subset of $C(Z)$, and hence $X$ is locally $Q$-complete.

(1) and (2) in Theorem 2 are generalized in the following form:

**Theorem 3.** *Suppose that $\mu$ is a complete structure of $X$ and $Y$ is a subspace of $X$, then $Y$ is open in $X$ if and only if $Y$ is locally complete with respect to the structure $\mu$.*

*Proof.* Suppose that $Y$ is open in $X$. For any point $y$ in $Y$ we take a neighborhood $U$ whose closure in $X$ is disjoint from $X-Y$. Then it is easily verified that $\overline{U}$(in $X$) is complete with respect to the structure $\mu$.

Conversely, if $B=X-Y$ is not closed, it is easily seen that there are a point $p$ in $Y$ such that $p\in(\overline{B}$(in $X)-B)$ and a neighborhood $V$(in $X$) of $p$ such that $\overline{V}$(in $Y$) is not complete with respect to the structure $\mu$, by the analogous method of the proof of $(1\to 2)$ in Theorem 2.

**Corollary.** *If $X$ is a $Q$-space, then any open subset of $X$ is locally $Q$-complete, and hence a local $Q$-space.*

Next we shall give the relations between the local compactness and local completeness.

**Theorem 4.** *The following conditions are equivalent:*

1) *$X$ is locally compact,*

2) *$X$ is open in any its $Q$-completion,*

3) *$X$ is locally complete with respect to the structure generated by any subset of $C(X)$.*

*Proof* ($1\rightarrow 2$). Let $Z$ be any $Q$-completion of $X$. Since $X$ is locally compact, for any point $x$ of $X$, there is a compact neighborhood of $x$ contained in $X$. On the other hand, any compact space has an only one structure which is complete. Therefore, $X$ is open in $Z$ by Theorem 3.

($1\rightarrow 2$). Let $\mu$ be a structure of $X$ generated by any subset of $C(X)$, and $Y$ be a completion of $X$ with respect to the structure $\mu$. Since $Y$ is complete with respect to $\mu$, $Y$ is a $Q$-space, and hence $Y$ is a $Q$-completion of $X$. By the assumption $X$ is open in $Y$. On the other hand, since $\mu$ is regarded as a complete structure of $Y$, $X$ is locally complete with respect to the structure $\mu$ by Theorem 3.

($3\rightarrow 1$). $\beta X$ is a $Q$-completion of $X$ with respect to the structure generated by $B(X)$. By the assumption, $X$ is locally complete with respect to the structure generated by $B(X)$, and hence, by Theorem 3, $X$ is open in $\beta X$. Therefore $X$ is locally compact.