# 47. *On the Spectral-resolutions of Quasi-compact Elements in a $B^*$-algebra*

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Let $\mathfrak{B}$ be a complex Banach algebra with unit element $e$. We shall consider the inverse of $\lambda e-a$ as a function of $\lambda$ for a fixed $a\in\mathfrak{B}$. According as $\lambda e-a$ is regular or singular in $\mathfrak{B}$, we say that $\lambda$ belongs to the resolvent set $\rho(a)$ or the spectrum $\sigma(a)$ of $a$. And a non-void set $\sigma$ is called a spectral set of $a$ if $\sigma$ is a subset of $\sigma(a)$ and $\sigma$ is both open and closed in $\sigma(a)$. For $\lambda$ in $\rho(a)$ the inverse of $\lambda e-a$ exists; it is denoted by $R(\lambda; a)$ and is called the resolvent of $a$. It is well known that the resolvent set $\rho(a)$ of $a$ is open and in each of its components $R(\lambda; a)$ is a regular function of $\lambda$. The following theorem is proved in [1, pp. 105-107 (Theorems 5.11.1 and 5.11.2)].

Theorem A. Let $\sigma(a)=\bigcup_{i=1}^{n}\sigma_i$ where each $\sigma_i$ is a spectral set of $a$ and $\sigma_i\frown\sigma_j=0$ when $i\neq j$. Let us suppose that closed Jordan curves $\Gamma_i$, $i=1, 2,\cdots, n$, satisfy the following conditions:

(i) For each $i(1\leqq i\leqq n)$ $\sigma_i$ is contained in the open domain $\mathfrak{D}_i$ which is bounded by $\Gamma_i$.

(ii) $\mathfrak{D}_i\frown\mathfrak{D}_j=0$ when $i\neq j$.

(iii) Each $\Gamma_i$ has the positive orientation, that is, the domain $\mathfrak{D}_i$ lies to the left of $\Gamma_i$.

If we define

$$(1)\qquad J_i=\frac{1}{2\pi i}\int_{\Gamma_i}R(\zeta; a)\,d\zeta \quad\text{and}\quad a_i=J_ia,\ i=1, 2,\cdots, n,$$

then

$$(2)\qquad \sum_{i=1}^{n}J_i=e,\quad J_i^2=J_i,\quad J_iJ_j=\theta,\ i\neq j,\quad J_i\neq\theta, e,\quad \sum_{i=1}^{n}a_i=a.$$

Furthermore, the spectrum $\sigma(a_i)$ of $a_i$ is $\sigma_i$ in addition to $\lambda=0$, that is, $\sigma(a_i)=\sigma_i\smile\{0\}$. In particular, if $\sigma_i$ is a single point $\lambda_i$, then an element $J_i(a-\lambda_ie)$ is quasi-nilpotent (or nilpotent).

First, under the assumption that $\mathfrak{B}$ is a commutative $B^*$-algebra (see Definition 1 below), we shall extend the above theorem to a case in which the spectrum $\sigma(a)$ of $a$ consists of infinitely many components. Next, by using this extension, we shall offer a new proof for the spectral resolution theorem of compact normal operators in Hilbert spaces.

Definition 1. A Banach algebra $\mathfrak{B}$ in which every element $a$ has an adjoint $a^*$ with $(\alpha a+\beta b)^*=\bar{\alpha}a^*+\bar{\beta}b^*$, $(ab)^*=b^*a^*$, $a^{**}=a$, and

$\|a^*a\|=\|a\|^2$ is called a $B^*$-algebra (see [1, p. 499]).

Lemma 1. If $a$ is an element in a $B^*$-algebra $\mathfrak{B}$ such that $a^*a=aa^*$, then we have $\lim_{n\to\infty}\|a^n\|^{1/n}=\|a\|$. Hence any commutative $B^*$-algebra has no quasi-nilpotent element except for $\theta$.

The proof is omitted.

Definition 2. Let $\mathfrak{B}$ be a Banach algebra with unit element $e$. An element $a\in\mathfrak{B}$ is called a quasi-compact element if the spectrum $\sigma(a)$ consists of finite points or an infinitely many points $\lambda_1, \lambda_2,\cdots,\lambda_n,\cdots$ such that $\lim_{n\to\infty}\lambda_n=0$.

Lemma 2. Let $\mathfrak{B}$ be a commutative $B^*$-algebra with unit element $e$ and $a$ an element in $\mathfrak{B}$. If $R(\lambda; a)$ is regular for $|\lambda|>R$, then we have $\|a\|\leqq R$.

Proof. Since $R(\lambda; a)=(\lambda e-a)^{-1}=\lambda^{-1}(e-a/\lambda)^{-1}$, we have easily

$$R(\lambda; a)=e\lambda^{-1}+\sum_{1}^{\infty}a^n\lambda^{-n-1} \quad \text{for } |\lambda|>\|a\|. \tag{3}$$

On the other hand, the right hand of the above equation is convergent for $|\lambda|>\lim_{n\to\infty}\|a^n\|^{1/n}$ and is divergent for $|\lambda|<\lim_{n\to\infty}\|a^n\|^{1/n}$. This implies that $\{\lambda; |\lambda|>R\}\subseteq\{\lambda; |\lambda|>\lim_{n\to\infty}\|a^n\|^{1/n}\}$, that is, $R\geqq\lim_{n\to\infty}\|a^n\|^{1/n}=\|a\|$ (see Lemma 1). The lemma is thereby proved.

Theorem B. Let $\mathfrak{B}$ be a commutative $B^*$-algebra with unit element $e$. If $a\in\mathfrak{B}$ is a quasi-compact element, $a$ is expressible in the form

$$a=\sum_i \lambda_i J_i \tag{4}$$

where $J_i$ is an idempotent element with $J_iJ_j=\theta$, $i\neq j$ and $\{\lambda_1, \lambda_2,\cdots\}$ is a finite set or an enumerable set such that $\lim_{n\to\infty}\lambda_n=0$.

Proof. Let $\{\lambda_1, \lambda_2,\cdots\}$ be the spectrum of $a$ except for 0. By the assumption of the lemma the set $\{\lambda_1, \lambda_2,\cdots\}$ is finite or enumerable (see Definition 2). Suppose that the spectrum $\sigma(a)$ is infinite. Without loss of generality we may assume that

$$|\lambda_1|\geqq|\lambda_2|\geqq\cdots\geqq|\lambda_n|\geqq\cdots\ . \tag{5}$$

We have clearly $\lim_{n\to\infty}|\lambda_n|=0$. Let $\Gamma_i$ be the circle of center $\lambda_i$ and radius $\varepsilon_i$, where $\varepsilon_i$ is a positive number such that

$$2\varepsilon_i<\mathrm{Min}\{\inf_{j\neq i}|\lambda_i-\lambda_j|,\ \inf_{|\lambda_j|\neq|\lambda_i|}|(|\lambda_i|-|\lambda_j|)|\}. \tag{6}$$

We define

$$J_i=\frac{1}{2\pi i}\int_{\Gamma_i}R(\zeta; a)\,d\zeta, \quad i=1,2,\cdots\ . \tag{7}$$

For an arbitrary number $\varepsilon>0$, there exists a natural number $N$ such that

$$|\lambda_N|\geqq\varepsilon, \quad |\lambda_{N+1}|<\varepsilon. \tag{8}$$

We set $\delta=(|\lambda_N|+|\lambda_{N+1}|)/2$ and let $\Gamma_\delta$ be the circle of center origin and radius $\delta$. We define

(9) $$J_\delta=\frac{1}{2\pi i}\int_{\Gamma_\delta} R(\zeta; a)\, ds.$$

In virtue of (6) we can easily see that

(10) $$\mathfrak{D}_\delta \frown \mathfrak{D}_i=0,\ i=1,2,\cdots,N \ \text{ and } \ \mathfrak{D}_i \frown \mathfrak{D}_j=0,\ i\neq j,$$

where $\mathfrak{D}_\delta$ and $\mathfrak{D}_i$ are open domains which are bounded by $\Gamma_\delta$ and $\Gamma_i$ respectively. Hence by Theorem A we have

(11) $$J_1+J_2+\cdots+J_N+J_\delta=e\ \left(a=\sum_{i=1}^{N}J_ia+J_\delta a\right),\ J_iJ_j=\theta,\ i\neq j,\ J_iJ_\delta=\theta$$

and

(12) $$\sigma(J_\delta a)=\{0,\ \lambda_{N+1},\ \lambda_{N+2},\cdots\}.$$

It follows from (12), (8) and Lemma 2 that $\|J_\delta a\|\leqq\varepsilon$. This implies that $a=\sum_{i=1}^{\infty}J_ia$. Furthermore, from Theorem A $\bar{a}_i=J_i(a-\lambda_ie)$ is quasi-nilpotent (or nilpotent). Hence by Lemma 1 $\bar{a}_i=J_i(a-\lambda_ie)=\theta$, that is, $J_ia=\lambda_iJ_i$. Consequently, we have $a=\sum_{i=1}^{\infty}\lambda_iJ_i$. When $\sigma(a)$ is finite, we have similarly $a=\sum_{i=1}^{n}\lambda_iJ_i$. Thus our theorem is completely proved.

Lemma 3. If $E$ is a bounded normal operator in a Hilbert space $\mathfrak{H}$ such that $E^2=E$, then $E$ is the projection of $\mathfrak{H}$ on some closed linear manifold $\mathfrak{M}$.

Proof. Let $H=EE^*(=E^*E)$. Then it is easily seen that $H^*=H$ and $H^2=H$. Hence $H$ is the projection of $\mathfrak{H}$ on some closed linear manifold $\mathfrak{M}$. We shall prove that $E=H$. For every $f\in\mathfrak{M}$ we have $f=EE^*f$ and hence $Ef=E(EE^*f)=E^2E^*f=EE^*f=f$. And for every $g\in\mathfrak{M}^\perp$ (orthogonal complement of $\mathfrak{M}$) we have $\|Eg\|^2=(Eg,Eg)=(E^*Eg,g)=(Hg,g)=(\theta,g)=0$, that is, $Eg=0$. Consequently, the operator $E$ coincides with the projection $H$ of $\mathfrak{H}$ on the closed linear manifold $\mathfrak{M}$.

Corollary to Theorem B. Let $H$ be a compact normal operator in a Hilbert space $\mathfrak{H}$. Then $H$ is expressed in the form

(13) $$H=\sum_i\lambda_iE_i$$

where $E_i$ is a projection such that $E_iE_j=\theta$, $i\neq j$, and $\{\lambda_1,\lambda_2,\cdots\}$ is a finite set or an enumerable set such that $\lim_{n\to\infty}\lambda_n=0$.

Proof. Let $\mathfrak{E}(\mathfrak{H})$ be the Banach algebra of endomorphisms of $\mathfrak{H}$. By corresponding to every $T\in\mathfrak{E}(\mathfrak{H})$, as the adjoint of $T$, the adjoint operator of $T$, $\mathfrak{E}(\mathfrak{H})$ becomes a $B^*$-algebra. Let $\{H,H^*\}'$ be the set of all the operators $T\in\mathfrak{E}(\mathfrak{H})$ such that $T$ commutes with both $H$ and $H^*$.

And let $\mathfrak{B}$ be the set of all the operators $T\in\mathfrak{E}(\mathfrak{H})$ such that $T$ commutes with every operator belonging to the set $\{H,H^*\}'$. Then we can easily see that $\mathfrak{B}$ is a commutative $B^*$-algebra containing $H$ and

$H^*$. Evidently any element in $\mathfrak{B}$ is a bounded normal operator in $\mathfrak{H}$. Since $H$ is a compact operator, we have from a theorem in Banach spaces (see [2, p. 166, Theorem 22]) the following:
(14) The set of all the proper values of the compact operator $H$ is a finite set or an enumerable set $\{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$ such that $\lim_{n\to\infty} \lambda_n = 0$. And it is not hard to show that the following three conditions are mutually equivalent:

(i) $\lambda I - H$ has a bounded inverse operator with domain $\mathfrak{H}$, where $I$ is the identity operator.

(ii) $\lambda I - H$ has, considered as an element in $\mathfrak{E}(\mathfrak{H})$, an inverse in $\mathfrak{E}(\mathfrak{H})$.

(iii) $\lambda I - H$ has, considered as an element in $\mathfrak{B}$, an inverse in $\mathfrak{B}$.
Hence from above and (14) we see that $H$ is, considered as an element in $\mathfrak{B}$, a quasi-compact element in $\mathfrak{B}$. According to Theorem B, $H$ is expressed in the form

$$H = \sum_i \lambda_i E_i$$

where $E_i$ is an idempotent element with $E_i E_j = \theta$, $i \neq j$ and $\{\lambda_1, \lambda_2, \cdots\}$ is a finite set or an enumerable set such that $\lim_{n\to\infty} \lambda_n = 0$. On the other hand it follows from Lemma 3 that $E_i$ is a projection in $\mathfrak{H}$. Hence the corollary is completely proved.

## References

[1] E. Hille: Functional Analysis and Semi-groups, New York (1946).
[2] S. Banach: Théorie des Opération Linéaires, Warszawa (1932).