59. On Ring Homomorphisms of a Ring of Continuous Functions

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Let γ be a linear subring of $C(X)$ and $H(\gamma)$ the totality of non-trivial ring homomorphisms[1] on γ and $H_0(\gamma)=H(\gamma)\smile\{\varphi_0\}$ where φ_0 denotes the trivial ring homomorphism, i.e. $\varphi_0=0$ on γ. We shall say that γ has the property (H) if the following property holds:

(H). *any* $\varphi\in H(\gamma)$ *is a point ring homomorphism* φ_x.[1] If $H(\gamma)$ is replaced by $H_0(\gamma)$, then we shall call that γ has the property (H_0). In case the property (H) or (H_0) holds respectively, if $\varphi_x\neq\varphi_y$ for $x\neq y$, then we say that γ has the property (H^*) or (H_0^*) respectively. Ishii [1] and Mrókwa [2] have obtained necessary and sufficient conditions that a subring γ containing constant functions has the property (H^*) or (H) respectively, under some conditions on γ. We denote by (h) one of the properties (H^*), (H), (H_0^*) and (H_0). In this paper we shall give a necessary and sufficient condition that $C(X)$ has a linear subring on which the property (h) is satisfied.[2] Moreover, we shall generalize Ishii's and Mrókwa's results, and give a weaker condition for which γ has the property (H).

1. Suppose that γ is a linear subring which has the property (h). Let us put

$$\hat{x}=\{y;\ f(x)=f(y) \text{ for all } f\in\gamma\}.$$

$\hat{x}$ is a closed subset of X because $\hat{x}=\bigcap_{f\in\gamma}f^{-1}(\alpha)$ where $f(x)=\alpha$. Then X is divided into a family $\hat{X}=\{\hat{x};\ x\in X\}$ of disjoint closed subsets of X. We shall define uniform neighborhoods of $\hat{x}$ by

$$W(f_1,\cdots,f_n;\varepsilon)(\hat{x})=\{\hat{y};\ |f_i(x)-f_i(y)|<\varepsilon,\ i=1,2,\cdots,n\}$$

where $f_i\in\gamma$ and $\varepsilon>0$. Then $\hat{X}$ becomes a uniform space with the uniform basis $\{W(f_1,\cdots,f_n;\varepsilon);f_i\in\gamma,\ \varepsilon>0\}$. In the following we denote by $Y=X/\gamma$ such a uniform space.

Let η be a mapping of X into Y defined by $\eta(x)=\hat{x}$ and η^* be a mapping of $C(Y)$ into $C(X)$ defined by $(\eta^*f)(x)=f(\eta(x))$ where $f\in C(Y)$ and $x\in X$. Then η is continuous and $\eta(x)=\eta(y)$ for any $y\in\hat{x}$ implies

1) A space X considered here is always a completely regular T_1-space, and other terminologies used here, for instance, $C(X)$, $B(X)$, ring homomorphisms and local Q-completeness, are the same as in [6, 7].

2) If we mean by $H(\gamma)$ the totality of non-trivial ring homomorphisms of γ *into* R, then we can replace by a subring a linear subring in Theorem 1 and Corollaries 1–5.

that $\eta^{*^{-1}}(\gamma)$ is a linear subring (written γ_Y) of $C(Y)$ and it is obvious that $\eta^*|\gamma_Y$ is one-to-one and γ_Y separates points of Y.

1) *If γ has the property (h), then γ_Y has the property (h).* Since $\varphi=\psi\eta^{*^{-1}}$ is a ring homomorphism of γ where ψ is a ring homomorphism of γ_Y, there exists a point x in X such that $\varphi=\varphi_x$. Therefore $\varphi_x(f)=f(x)$ implies that, for any $g\in\gamma_Y$, $\psi(g)=\psi(\eta^{*^{-1}}\eta^*(g))=(\psi\eta^{*^{-1}})(\eta^*(g))=(\eta^*(g))(x)=g(\eta(x))=g(\hat{x})=\psi_{\hat{x}}(g)$.

Any $f\in\gamma_Y$ has a continuous extension $\tilde{f}$ over νY. In the following we denote $\tilde{f}$ by such a continuous extension of f over νY. If there exists a point p in $\nu Y-Y$, $\varphi_p(\tilde{f})=\tilde{f}(p)$ is a ring homomorphism on γ_Y since γ_Y is linear.[3] Hence either there is a point x in Y such that $\varphi_p=\varphi_x$ (not necessarily $\varphi_x\neq 0$) or $\varphi_p=0$ on γ_Y. Let us put $B=\{p;\ \varphi_p=0,\ p\in\nu Y\}$ and $\tilde{x}=\{y;\ \varphi_y=\varphi_x,\ y\in\nu Y$ and $\varphi_x\neq 0\}$. Then B is a closed subset of νY because $B=\bigcap_{f\in\gamma_Y}\tilde{f}^{-1}(0)$. Since γ_Y separates the point of Y, $\tilde{x}\frown Y=\{x\}$ for each point $x\in Y$. We have moreover the following

2) *$\tilde{x}=\{x\}$ for each point x in Y.* Suppose that $\tilde{x}\ni p\in\nu Y-Y$. Let $\{a_\alpha;\ \alpha\in Y\}$ be a directed set which converges to p and $a_\alpha\in Y$ for each α. For any $f_1,\cdots,f_n\in\gamma_Y$, $\varepsilon>0$, there exists an index α_0 such that $W(\tilde{f}_1,\cdots,\tilde{f}_n;\varepsilon)(p)\ni a_\alpha$ for $\alpha>\alpha_0$. By the assumption $\tilde{x}\ni p$, we have $W(f_1,\cdots,f_n;\varepsilon)(x)\ni a_\alpha$ for $\alpha>\alpha_0$. This means $\{a_\alpha;\ \alpha\in p\}$ converges to x, that is, $x=p$. This is a contradiction.

From 2) we have

3) *$\nu Y=Y\smile B$, $Y\frown B=\theta$, that is, Y is open in νY, in other words, Y is locally Q-complete* (see [6]).

4) *If $B=\theta$, then Y is a complete uniform space and hence Y is a Q-space.* For the structure of Y is generated by a subset γ_Y of $C(Y)$.

5) *If γ contains constant functins, then $B=\theta$.* For if γ contains constant functions, then $\varphi_p\neq 0$ for any $p\in\nu Y$.

6) *If γ has the property (H_0), then $B=\theta$.*

2. Theorem 1. *$C(X)$ has a linear subring on which the property (H) holds if and only if there exists a locally Q-complete space Y which is a continuous image of X.*

Proof. If $C(X)$ contains a subring γ on which the property (H) holds, then by 3), $Y=X/\gamma$ is locally Q-complete. Conversely if X has a continuous image Y which is locally Q-complete, then $C_B(Y)$, where B is a closure of $\nu Y-Y$ in βY, has the property (H^*) by Theorem 1 in [7]. On the other hand, $C_B(Y)$ can be considered as a linear subring of $C(X)$, and hence $C(X)$ has a subring on which the property

3) If γ_Y is not linear, $\varphi_p(\gamma_Y)$ may be a proper subring of R.

(H) holds.

In Theorem 1, if Y is not a Q-space, then we have a Q-space which is a continuous image of Y by Theorem 1 in [6]. Conversely, for any Q-space Z, it is well known that $C(Z)$ has the property (H^*) [3, 4]. Therefore we can state Theorem 1 as follows: $C(X)$ has a linear subring on which (H) holds if and only if there exists a Q-space which is a continuous image of X. For the properties (H_0) or (H_0^*), it is an open question, in Theorem 1, whether (H) is replaced by (H_0) or not. But we notice that if $C(X)$ has the property (H_0), Y is a Q-space, conversely if Y is a Q-space which contains a subset Y_1 such that $\nu Y_1 = Y$ or Y is a one-point Q-completion of Y_1, then $C(X)$ has a linear subring on which the property (H_0) holds [7, 8].

The following corollaries are an immediate consequences of Theorem 1.

Corollary 1. *$C(X)$ has a linear subring on which the property (H^*) holds if and only if there is a locally Q-complete space which is a one-to-one continuous image of X.*

Corollary 2. *$C(X)$ has a linear subring γ such that γ has the property (H^*) and γ generates a structure of X if and only if X is locally Q-space.*

In Corollary 2, if γ has constant functions, X is always a Q-space. If γ has no constant functions, the X is not necessarily a Q-space. Such an example is given in [7].

If γ is a subring of $B(X)$, then we have the following corollaries replacing νY by βY in the arguments in § 1.

Corollary 3. *$B(X)$ has a linear subring on which the property (H) or (H_0) if and only if there is a locally compact space Y which is a continuous image of X.*

In this case it is easily seen that Y is replaced by a compact space Z. Such a compact space Z is given by a space which is obtained from βY by contracting $\beta Y - Y$ to a point in Y. The next corollaries 4 and 5 follow from Corollaries 2 and 3.

Corollary 4. *$B(X)$ has a linear subring on which the property (H^*) or (H_0^*) if and only if there exists a locally compact space which is a one-to-one continuous image of X.*

Corollary 5. *$B(X)$ has a linear subring γ such that γ has the property (H^*) or (H_0^*) and γ generates a structure of X if and only if X is a locally compact space.*

In Corollary 5, if either γ contains constant functions or has the property (H_0^*), then X is always compact by 5) and 6).

3. In this section, we assume that γ contains constant functions, and we shall give, under some conditions, a necessary and sufficient condition that γ has the property (H).

Theorem 2. *Let γ be a subring of $C(X)$ which satisfies the following conditions*:

1) *γ contains constant functions,*

2) *if $\gamma \ni f$, $f \geq \alpha > 0$, then $\gamma \ni 1/f$.*

Then γ has the property (H) if and only if a uniform space $Y = X/\gamma$ is complete.

Proof. Necessity follows from Theorem 1 and 5) in § 1. Sufficiency follows immediately from the proof of sufficiency of Theorem 1 in [1] if we replace γ and X by γ_Y and Y respectively, and we notice that Y is a complete uniform space.

Theorem 2 is a generalization of Mrókwa's Theorem 2 in [2] and the condition on γ is weaker than Mrókwa conditions on γ. The following theorem is obtained by Mrókwa (Theorem 1 in [2]) in the another form:

Theorem 3 (*Mrókwa*). *Let γ be a subring of $B(X)$ which satisfies the following conditions*:

1) *γ contains constant functions,*

2) *γ is uniformly closed.*

Then γ has the property (H) if and only if $Y = X/\gamma$ is compact.

Proof. Necessity follows from the note in the end of § 2. Conversely, suppose that Y is a compact space. Since γ contains constant functions and is uniformly closed, γ_Y also contains constant functions and is uniformly closed. Therefore, γ_Y coincides with $C(Y)$ by Stone's theorem [5]. On the other hand, $C(Y)$ is considered as a subring C_1 of $C(X)$ and it is easy to see that C_1 coincides with γ.

Finally we shall prove the following theorem which is the first part of Theorem 3 in [3] as an application of Theorem 2.

Theorem 4 (*Mrókwa*). *Let X be a Lindelöf space. If γ is a subring of $C(X)$ which satisfies the following conditions*:

1) *γ contains constant functions,*

2) *γ is uniformly closed,*

3) *if $\gamma \ni f > 0$, then $1/f \in \gamma$.*

Then γ has the property (H).

Proof. Let $Y = X/\gamma$ and $\tilde{Y}$ be a completion of a uniform space Y. If we replace νY by $\tilde{Y}$ in the arguments in § 1, then it is easy to see, using the same method used as in the proof of 2) in § 1, that B consists of only one point p, that is, $\tilde{Y} = Y \cup \{p\}$. It is obvious that γ_Y satisfies the conditions (1), (2) and (3), and any $f \in \gamma_Y$ has a continuous extension over $\tilde{Y}$. Let us put $\tilde{\gamma} = \{\tilde{f}; f \in \gamma\}$. Then $\tilde{\gamma}$ is a subring of $C(\tilde{Y})$ and $\tilde{\gamma}$ has the properties (1) and (2) in Theorem 2. Since $\tilde{Y}$ has the complete structure generated by $\tilde{\gamma}$, $\tilde{\gamma}$ has the property (H) by Theorem 2. It is therefore sufficient to prove theorem that $Y = \tilde{Y}$.

For this purpose, we shall show that the existence of the point p leads a contradiction. Since γ_Y separates points of Y and $\widetilde{f}(p)=0$ for all $f\in\gamma_Y$, there is a function $f_y\in\gamma_Y$ for any point y in Y such that $f_y(y)\neq 0$. Let us put $U(y;f_y)=\{x;\,|f_y^2(x)-f_y^2(y)|<f_y^2(y)/2\}$. Then $\{U(y;f_y);\ y\in Y\}$ forms an open covering of Y. Since Y is a continuous image of X, Y is a Lindelöf space, and hence there is a countable subcovering $\{U(y_n;f_{y_n});\ n=1,2,\cdots\}$. Let us put $f_i=f_{y_i}$ and

$$g_n=\sum_{i=1}^{n}\frac{1}{2^i}\frac{f_i^2}{1+f_i^2}.$$

Then $\{g_n;\ n=1,2,\cdots\}$ converges uniformly to $g=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{f_i^2}{1+f_i^2}$. Since γ_Y satisfies the condition 2), we have $g\in\gamma_Y$. By the methods of construction of g_n, g is positive on Y and $\widetilde{g}(p)=0$. But the positiveness of g on Y implies that $1/g\in\gamma_Y$ by the property 3). On the other hand, $1/g$ has not a continuous extension over $\widetilde{Y}$. This is a contradiction. Therefore we have $Y=\widetilde{Y}$.

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