72. *On the Singular Integrals. VI*[*)]

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1. We begin with the following

Definition 1. *By* W_2 *we denote the class of functions which are measurable over* $(-\infty, \infty)$ *and satisfy*

$$\int_{-\infty}^{\infty}\frac{|f(t)|^2}{1+t^2}dt<\infty. \tag{1.01}$$

For this class, the generalized Hilbert transform of order 1 is precisely corresponding. This modified one is defined as follows [4, V]:

$$\tilde{f}_1(x)=\frac{x+i}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\frac{dt}{x-t}. \tag{1.02}$$

The main purpose of this chapter is to determine the relation of spectrum between any given function $f(x)$ of the class W_2 and its generalized Hilbert transform of order 1. We shall quote the Plancherel theorem of Fourier transform repeatedly [2]. We introduce the generalized Fourier transform due to N. Wiener [6]. This is defined by

$$s^f(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}f(x)\frac{e^{-iux}-1}{-ix}dx + \underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_{-A}^{-1}+\int_{1}^{A}\right]f(x)\frac{e^{-iux}}{-ix}dx. \tag{1.03}$$

Then by the Plancherel theorem, the Fourier-Wiener transform $s^f(u)$ is well defined and

$$s^f(u+\varepsilon)-s^f(u-\varepsilon)=\underset{A\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(t)\frac{2\sin\varepsilon t}{t}e^{-iut}dt, \tag{1.04}$$

$$\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2du=\frac{1}{\pi\varepsilon}\int_{-\infty}^{\infty}|f(t)|^2\frac{\sin^2\varepsilon t}{t^2}dt. \tag{1.05}$$

If $f(x)$ belongs to the class W_2, then by Theorem 1 of [4, V] the Fourier-Wiener transform of $\tilde{f}_1(x)$ is also defined. We will denote this by $\tilde{s}^f_1(u)$.

Throughout this paper, let $g(x)$ be a real valued measurable function which belongs to the class W_2. We also denote

$$f_1(x)=g(x)+i\tilde{g}_1(x). \tag{1.06}$$

We shall prove the following fundamental

Theorem 1. *Let* $g(x)$ *belong to the class* W_2. *Then for any given positive number* ε,

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*) Details will appear in Jour. Fac. Sci., Hokkaidô Univ.

(i) *if* $|u|>\varepsilon$, *then*

(1.07) $$\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)=(-i\ \mathrm{sign}\ u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}$$

and

(ii) *if* $|u|\leqq\varepsilon$, *then*

(1.08) $$\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)=i\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}+2r_1^g(u+\varepsilon)+2r_2^g(u+\varepsilon),$$

where

(1.09) $$r_1^g(u)=\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,\frac{e^{-ius}-1}{-is}\,ds$$

(1.10) $$r_2^g(u)=\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,e^{-ius}ds.$$

We remark that in (1.09) and (1.10), the limit operation is taken over $(-\infty,\infty)$.

Theorem 2. *Under the assumption of Theorem* 1, *we have for any given positive number* ε,

(i) *if* $|u|>\varepsilon$, *then*

(1.11) $$s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)=(1+\mathrm{sign}\ u)\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}$$

and

(ii) *if* $|u|\leqq\varepsilon$, *then*

(1.12) $$s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)=2ir_1^g(u+\varepsilon)+2ir_2^g(u+\varepsilon),$$

where $r_1^g(u)$ *and* $r_2^g(u)$ *are defined by* (1.09) *and* (1.10) *respectively.*

2. We now introduce the following class of functions:

Definition 2. *By* S_0 *we denote the class of functions such that*

(2.01) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\,dt \qquad \textit{exists.}$$

Then we can prove

Theorem 3. *Let* $g(x)$ *be a real valued measurable function of the class* S_0. *Let us assume that*

(K_1) $$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\ du=0$$

and

(K_2) *there exists a constant* a^g *such that*

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{0}^{2\varepsilon}\left|\underset{B\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}\,e^{-ius}ds-\sqrt{\frac{\pi}{2}}\,a^g\right|^2du=0.$$

Then its generalized Hilbert transform of order 1, $\tilde{g}_1(x)$ *belongs, also to the same class* S_0 *and*

(2.02) $$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\tilde{g}_1(t)|^2\,dt=|a^g|^2+\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt.$$

For the proof of this theorem we quote the following theorem which is called usually the Wiener formula [3, 5]:

Theorem A. *If $f(x)\geqq 0$ for $0<x<\infty$, and either limits*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)dt \tag{2.03}$$

or

$$\lim_{\varepsilon\to 0}\frac{2}{\pi\varepsilon}\int_0^\infty f(t)\,\frac{\sin^2\varepsilon t}{t^2}\,dt \tag{2.04}$$

exists, then the other limit exists and assumes the same value.

From this theorem, the Plancherel theorem and (1.05), we get

Theorem B. *Let $f(x)$ be a measurable function for which* (2.01) *is bounded in T, $0<T<\infty$. Then we have*

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2du=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2\,dt \tag{2.05}$$

in the sense that if eithr side of two limits exists, the other limit does and assumes the same value.

Therefore if we prove

Theorem 4. *Under the assumption of Theorem* 3, *we have*

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|\tilde{s}_1^g(u+\varepsilon)-\tilde{s}_1^g(u-\varepsilon)|^2\,du \tag{2.06}$$

$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du.$$

Then we get Theorem 3 immediately. By the same argument we get

Theorem 5. *Under the assumption of Theorem* 3, *$f_1(x)$ defined by* (1.06), *belongs to the same class S_0 and we have*

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi\varepsilon}\int_{-\infty}^{\infty}|s_1^f(u+\varepsilon)-s_1^f(u-\varepsilon)|^2\,du \tag{2.07}$$

$$=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty}|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du$$

and

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f_1(t)|^2\,dt \tag{2.08}$$

$$=|a^g|^2+2\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|g(t)|^2\,dt.$$

3. We consider now functions of classes S and S' which have been introduced by N. Wiener [6].

Definition 3. *By S we denote the class of functions such that*

$$\varphi^f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x+t)\overline{f(t)}\,dt \tag{3.01}$$

exists for every x.

Definition 4. *By S' we denote the class of functions such that $\varphi^f(x)$ defined by* (3.01) *exists for every x and continuous over* $(-\infty, \infty)$.

It is clear that

$$S' \subset S \subset S_0. \tag{3.02}$$

Then we shall prove

Theorem 6. *Let $g(x)$ be a real valued measurable function of the class S. Let us assume that the conditions* (K_1) *and* (K_2) *of Theorem* 3 *are satisfied. Then its generalized Hilbert transform of order* 1, $\tilde{g}_1(x)$ *belongs also to the same class S and if we denote*

$$\tilde{\varphi}_1^g(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\tilde{g}_1(t+x)\,\overline{\tilde{g}_1(t)}\,dt, \tag{3.03}$$

then

$$\tilde{\varphi}_1^g(x)=|a^g|^2+\varphi^g(x) \tag{3.04}$$

and

$$\tilde{\varphi}_1^g(x)=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^{\infty}\cos ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du. \tag{3.05}$$

Theorem 7. *Under the assumption of Theorem* 5 except *that $g(x)$ belongs to the class S', $\tilde{g}_1(x)$ belongs also to the same class and* (3.04), (3.05) *are true.*

Theorem 8. *Under the assumption of Theorem* 6, *the necessary and sufficient condition that $f_1(x)$ defined by* (1.06) *belongs to the class S, is that*

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^{\infty}\sin ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du \tag{3.06}$$

exists for every x.

In this case if we denote

$$\varphi_1^f(x)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_1(t+x)\overline{f_1(t)}\,dt, \tag{3.07}$$

then

$$\varphi_1^f(x)=|a^g|^2+\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\int_0^{\infty}e^{iux}\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du. \tag{3.08}$$

Theorem 9. *Under the assumption of Theorem* 7, *the necessary and sufficient condition that $f_1(x)$ defined by* (1.06) *belongs to the class S', is that*

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi\varepsilon}\int_0^{\infty}\sin ux\,|s^g(u+\varepsilon)-s^g(u-\varepsilon)|^2\,du \tag{3.06}$$

exists for every x and is continuous over $(-\infty, \infty)$.

On the other hand, N. Wiener [7] has also proved the following two theorems:

Theorem C. *If $f(x)$ belongs to S and $\varphi^f(x)$ defined by* (3.01) *is continuous at point $x=0$, then it is continuous for all real arguments and $f(x)$ belongs to S'.*

Theorem D. *If $f(x)$ belongs to S, it will belong to S' when and only when*

$$\text{(3.09)} \qquad \lim_{A\to\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{4\pi\varepsilon}\left[\int_{-\infty}^{-A}+\int_{A}^{\infty}\right]|s^f(u+\varepsilon)-s^f(u-\varepsilon)|^2\,du=0.$$

From these theorems we have immediately

Theorem 10. *Under the assumption of Theorem* 7, *if* $f_1(x)$ *defined by* (1.06) *belongs to the class* S, *then it belongs necessarily to the class* S'.

4. We can apply the result of the preceding sections to almost periodic functions. Here we consider almost periodic functions in a sense of Besicovitch [1].

Theorem 11. *Let* $g(x)$ *be a real valued measurable function over* $(-\infty, \infty)$. *Let* $g(x)$ *be a* B_2-*almost periodic function. Let us assume that the condition* (K_1) *is satisfied. Then the necessary and sufficient condition for the generalized Hilbert transform* $\tilde{g}_1(x)$ *to be also* B_2-*almost periodic is that the condition* (K_2) *is satisfied for* a^g—*the constant term of* $\tilde{g}_1(x)$. *If the associated Fourier series with* $g(x)$ *is*

$$\text{(4.01)} \qquad g(x)\sim{\sum}' a_n e^{i\lambda_n x},$$

then

$$\text{(4.02)} \qquad \tilde{g}_1(x)\sim a^g+{\sum}'(-i\operatorname{sign}\lambda_n)\,a_n e^{i\lambda_n x},$$

where the prime means that the summation does not contain the constant term.

Theorem 12. *Under the assumption of Theoren* 11, *the necessary and sufficient condition for* $f_1(x)=g(x)+i\tilde{g}_1(x)$ *to be* B_2-*almost periodic is that the condition* (K_2) *is satisfied for* a^g—*the constant term of* $\tilde{g}_1(x)$. *The associated Fourier series is*

$$\text{(4.03)} \qquad f_1(x)\sim ia^g+2\sum_{\lambda_n>0}{}' a_n e^{i\lambda_n x}.$$

5. For the class W_2, the Hilbert transform in ordinary sense does not necessarily exist. However from the identity

$$\text{(5.01)} \qquad \tilde{f}_1(x)=\tilde{f}(x)+A^f$$

where

$$\text{(5.02)} \qquad A^f=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t+i}\,dt,$$

it is equivalent that the constant term A^f is finitely determined. Therefore from properties of $\tilde{f}_1(x)$, those of $\tilde{f}(x)$ may be deduced. In this case the condition (K_2) may be replaced by

$$(\mathrm{K}_3) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_0^{2\varepsilon}\left|\underset{B\to\infty}{\text{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-B}^{B}\frac{g(s)}{s+i}e^{-ius}\,ds-\sqrt{\frac{\pi}{2}}A^g\right|^2 du=0.$$

We omit details here. We will end this paper by adding some remarks:

Remark 1. In Theorem 11, from (K_1) we get

$$(\mathrm{K}_1) \qquad \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\{s^g(u+\varepsilon)-s^g(u-\varepsilon)\}\,du=0.$$

From this and the theorem of Bochner-Hardy-Wiener [3, 5], we get

$$(c_1) \qquad a_0=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}g(t)\,dt=0.$$

Conversely if we assume (c_1), then (K_1) is deduced by the aid of Bochner's representation theorem of a positive definite function. Therefore our assumption (K_1) does not mean the loss of generality for almost periodic functions.

Remark 2. As in Theorem 11 let $g(x)$ be B_2-almost periodic and (K_1) is satisfied. If we assume that

$$(5.03) \qquad \sum_{\lambda_n<0}|a_n e^{\lambda_n}|<\infty,$$

then (K_2) is equivalent to the following relation:

$$(c_2) \qquad \lim_{p\to\infty}|a^g-a^\sigma|=0,$$

where

$$(5.04) \qquad a^\sigma=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma^g_{B_p}(t)}{t+i}\,dt=-2i\sum_{\lambda_n<0}d^B_n a_n e^{\lambda_n},$$

$$(5.05) \qquad \sigma^g_{B_p}(x)=\sigma^g_{\binom{n_1,n_2,\cdots,n_p}{\beta_1,\beta_2,\cdots,\beta_p}}(x)$$

$$=\sum\left(1-\frac{|\nu_1|}{n_1}\right)\cdots\left(1-\frac{|\nu_p|}{n_p}\right)a_n e^{i\lambda_n x}$$

and

$$(5.06) \qquad \lambda_n=\frac{\nu_1}{n_1}\beta_1+\cdots+\frac{\nu_p}{n_p}\beta_p.$$

In particular if

$$(5.07) \qquad \underset{\lambda_m,\lambda_n<0}{\text{g.l.b.}}|\lambda_m-\lambda_n|>0$$

or

$$(5.08) \qquad \lambda_n=-\log(|n|+1), \qquad n=-1,-2,\cdots,$$

then the condition (5.03) is satisfied.

References

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