96. Some Characterizations of Fourier Transforms

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In the following we shall show that the Fourier cosine transform and the Fourier exponential transform are characterized by some of their properties.

At first we shall prove a number-theoretical lemma. Let

$$p_1 < p_2 < p_3 < \cdots$$

be the all prime numbers and $\mu_\nu(n)$ a function defined at every natural number such that $\mu_\nu(n)=\mu(n)$, if every prime divisor of n is one of $p_1, p_2, \cdots, p_\nu$, and $\mu_\nu(n)=0$ otherwise.

Lemma. *Let $f(n)$ be a function defined at every non-negative integer and $\sum_{n=0}^{\infty} f(n)$ absolutely convergent. Let us denote*

$$F(m)=\sum_{n=0}^{\infty} f(mn)$$

for every natural number m. Then

$$f_\nu(m)=\sum_{n=1}^{\infty} \mu_\nu(n)\ F(mn)$$

converges to $f(m)$ as $\nu \to \infty$.

Proof. We have

$$f_\nu(m)=\sum_{n=0}^{\infty} f(mn)\sum_{d \mid n} \mu_\nu(d)$$

and

$$\sum_{d \mid n} \mu_\nu(d)=\begin{cases} 1, & (n, p_1 p_2 \cdots p_\nu)=1, \\ 0 & \text{otherwise,} \end{cases}$$

therefore

$$f_\nu(m)=\sum f(mn),$$

where n ranges over all positive integers prime to $p_1 p_2 \cdots p_\nu$. Then

$$| f(m)-f_\nu(m) | \leqq \textstyle\sum_{n>p_\nu} | f(mn) |$$

and the right hand side of this inequality tends to 0 as $\nu \to \infty$. Q. E. D.

By $\mathfrak{D}$ we denote the family of all $C^\infty(R)$-functions with compact carrier. For a given continuous function $F(x)$ we denote

$$F\varphi(x)=\int_{-\infty}^{\infty} F(xt)\varphi(t)dt, \qquad \varphi \in \mathfrak{D}.$$

Theorem 1. *Let an even function $C(x)$ be the second derivative of a bounded function, and*

$$\sum_{n=-\infty}^{\infty} C\varphi(n)=\sum_{n=-\infty}^{\infty} \varphi(n) \tag{1}$$

for all $\varphi \in \mathfrak{D}$. Then

$$C(x)=\cos 2\pi x.$$

Proof. From the Poisson summation formula and (1) we get

$$H\varphi(0)+2\sum_{n=1}^{\infty} H\varphi(n)=0,$$

where

$$H(x)=C(x)-\cos 2\pi x.$$

By the hypotheses of the theorem there is a bounded function $G(x)$ such that $G''(x)=H(x)$, so

$$\begin{aligned} H\varphi(n) &= \int_{-\infty}^{\infty} H(nx)\varphi(x)\,dx \\ &= \frac{1}{n^2}\int_{-\infty}^{\infty} G(nx)\varphi''(x)dx \\ &= O\left(\frac{1}{n^2}\right). \end{aligned}$$

Hence $\sum_{n=1}^{\infty} H\varphi(n)$ is absolutely convergent. If $\varphi(x)\in\mathfrak{D}$, then $\varphi_m(x)=\varphi(x/m)\in\mathfrak{D}$. Therefore

$$H\varphi_m(0)+2\sum_{n=1}^{\infty} H\varphi_m(n)=0. \tag{2}$$

But we have

$$\begin{aligned} H\varphi_m(x) &= \int_{-\infty}^{\infty} H(xt)\varphi\left(\frac{t}{m}\right)dt \\ &= \int_{-\infty}^{\infty} H(mxt)\varphi(t)\,|\,m\,|\,dt \qquad (3) \\ &= |\,m\,|\,H\varphi(mx). \end{aligned}$$

By (2) and (3)

$$H\varphi(m0)+2\sum_{n=1}^{\infty} H\varphi(mn)=0.$$

Applying our lemma we get

$$2H\varphi(1)=\lim_{\nu\to\infty}\sum_{n=1}^{\infty}\mu_\nu(n)\cdot 0=0,$$

which means

$$\int_{-\infty}^{\infty} H(x)\varphi(x)\,dx=0$$

for every $\varphi\in\mathfrak{D}$; that is,

$$H(x)\equiv 0. \qquad \text{Q. E. D.}$$

Next we want to deal with Fourier exponential transforms:

Theorem 2. *Let $E(x)$ be a bounded continuous function of a real variable and not equal to the constant* 0. *If*

$$E(\varphi*\psi)(x)=E\varphi\cdot E\psi \tag{1}$$

for every pair of functions $\varphi,\psi\in\mathfrak{D}$, and

$$\sum_{n=-\infty}^{\infty} E\varphi(n)=\sum_{n=-\infty}^{\infty}\varphi(n), \tag{2}$$

then

$$E(x)=e^{2\pi ix} \qquad or \qquad e^{-2\pi ix}.$$

Proof. From now we denote $\varphi_h(x)=\varphi(x-h)$. Because of $E(x)\not\equiv 0$ there exists a function $\varphi\in\mathfrak{D}$ such that

$$E\varphi(1)\neq 0.$$

Let us denote

$$B(h)=\frac{E\varphi_h(1)}{E\varphi(1)}.$$

For every $\varphi\in\mathfrak{D}$ we have

$$\begin{aligned}\psi_h*\varphi&=\int_{-\infty}^{\infty}\psi(x-t+h)\varphi(t)\,dt\\&=\int_{-\infty}^{\infty}\psi(x-t)\varphi(t-h)\,dt\\&=\psi*\varphi_h\end{aligned}$$

therefore

$$E\psi_h\cdot E\varphi=E(\psi_h*\varphi)=E(\psi*\varphi_h)=E\psi\cdot E\varphi_h,$$

and hence

$$E\psi_h(1)=B(h)\cdot E\psi(1). \tag{3}$$

Setting $\psi=\varphi_k$ into (3) we obtain

$$E\varphi_{k+h}(1)=B(h)E\varphi_k(1)$$

and it follows from (3) that

$$B(h+k)E\varphi(1)=B(h)B(k)E\varphi(1).$$

Since $E\varphi(1)\neq 0$ we get

$$B(h+k)=B(h)B(k);$$

and we can denote

$$B(h)=e^{ibh}$$

with a complex constant b. (It is impossible that B is 0.) Now we shall transform the formula (3):

$$\begin{aligned}E\psi_h(1)&=\int_{-\infty}^{\infty}E(t)\psi(t-h)\,dt\\&=\int_{-\infty}^{\infty}E(t+h)\psi(t)\,dt\end{aligned}$$

and

$$B(h)\cdot E\psi(1)=\int_{-\infty}^{\infty}e^{ibh}E(t)\psi(t)\,dt;$$

hence

$$\int_{-\infty}^{\infty}(E(t+h)-e^{ibh}E(t))\psi(t)\,dt=0$$

for all functions $\psi\in\mathfrak{D}$. So we get

$$E(x+h)=E(x)e^{ibh}$$

and

$$E(h)=ce^{ibh}, \qquad c=E(0)\neq 0.$$

By the boundedness of $E(x)$ b must be a real number. Thus we have

$$E\varphi(x)=\int_{-\infty}^{\infty} ce^{ibxt}\varphi(t)\,dt=cF\varphi\Big(\frac{b}{2\pi}x\Big),$$

where

$$F\varphi(x)=\int_{-\infty}^{\infty} e^{2\pi ixt}\varphi(t)\,dt.$$

Therefore

$$\sum_{n=-\infty}^{\infty} E\varphi(n)=c\sum_{n=-\infty}^{\infty} F\varphi\left(\frac{b}{2\pi}n\right),$$

which is by the Poisson summation formula equal to

$$c\sum_{n=-\infty}^{\infty}\left|\frac{2\pi}{b}\right|\varphi\left(\frac{2\pi}{b}n\right). \tag{4}$$

But by the hypothesis (2) of the theorem it is also equal to

$$\sum_{n=-\infty}^{\infty}\varphi(n). \tag{5}$$

If $2\pi>|b|$, we take as φ such a function $\in\mathfrak{D}$ that its carrier contains neither n nor $\frac{2\pi}{b}n$ with the exception 1 and $\varphi(1)\neq 0$. To such φ (4) is not equal to (5). So it must be

$$2\pi\leqq|b|.$$

Similarly $2\pi\geqq|b|$. And finally we get

$$2\pi=|b|, \quad c=1. \qquad \text{Q. E. D.}$$

To any valuation vector ring we can prove a result similar to Theorem 2. (In this case we must consider $\mathfrak{D}^0$.)

References

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