95. Remarks on a Theorem concerning Conformal Transformations

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In a recent paper K. Yano and T. Nagano [3] proved the following

Theorem A. *Let M be a complete Einstein manifold and suppose that a vector field on M generates globally a one-parameter group of non-homothetic conformal transformations.*[1] *Then M is isometric to a spherical space, i.e. a simply connected, complete space of positive constant sectional curvature. In particular M is homeomorphic to the sphere S^n.*

On the other hand, S. Ishihara and the present author [1] investigated the topological and differential-geometrical properties of compact or complete Riemannian manifolds admitting a concircular transformation. A concircular transformation of a Riemannian manifold M with metric $g_{\mu\lambda}$ into a Riemannian manifold $'M$ with metric $'g_{\mu\lambda}$ is by definition a conformal transformation

$$'g_{\mu\lambda}=\rho^2 g_{\mu\lambda}, \tag{1}$$

which carries geodesic circles in M to geodesic circles in $'M$, and is characterized by the equation

$$\nabla_\mu \rho_\lambda - \rho_\mu \rho_\lambda = \psi g_{\mu\lambda}, \tag{2}$$

where ρ is a positive-valued function on M, $\rho_\lambda = \nabla_\lambda \log \rho$ and ψ is a function on M. We obtained the following theorems:

Theorem B. *Let M and $'M$ be Riemannian manifolds whose scalar curvatures k and $'k$ are constant. We assume that M is complete and that there exists a concircular transformation of M into $'M$. Then the manifold M is*

I) *a Euclidean space, if $k=0$,*

II) *a spherical space, if $k>0$, or*

III) *a hyperbolic space, if $k<0$.*[2]

Theorem C. *In addition to the assumptions of Theorem* B, *assume that $'M$ is complete too and the concircular transformation*

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1) In this paper we suppose that manifolds are always connected, of dimension $n>2$ and of class C^∞, and that the differentiability of transformations and quantities is also of class C^∞. Greek indices run from 1 to n. We shall deal only with non-homothetic conformal transformations, and the term "conformal" will always mean "non-homothetic conformal".

2) The scalar curvatures in this paper are different from those in [7] in the sign.

is a homeomorphism of M onto 'M. Then the scalar curvatures k and 'k should be positive and both M and 'M are spherical spaces.

First we notice the following

Theorem 1. *In order that a conformal transformation map an Einstein manifold into an Einstein one, it is necessary and sufficient that the transformation be concircular.*

Proof. The sufficiency was given by K. Yano [2]. For a conformal transformation (1), it is well known that we have

$$'K_{\nu\mu\lambda}{}^{\kappa}=K_{\nu\mu\lambda}{}^{\kappa}-A^{\kappa}_{\nu}\rho_{\mu\lambda}+A^{\kappa}_{\mu}\rho_{\nu\lambda}-\rho_{\nu}{}^{\kappa}g_{\mu\lambda}+\rho_{\mu}{}^{\kappa}g_{\nu\lambda}, \tag{3}$$

$$'K_{\mu\lambda}=K_{\mu\lambda}-(n-2)\rho_{\mu\lambda}-g_{\mu\lambda}\rho_{\kappa}{}^{\kappa}, \tag{4}$$

$$n'k\rho^2=nk-2\rho_{\kappa}{}^{\kappa}, \tag{5}$$

where $K_{\nu\mu\lambda}{}^{\kappa}$, $K_{\mu\lambda}$ and k are the curvature tensor, the Ricci tensor and the scalar curvature of M respectively, the prime indicates the corresponding quantities of $'M$ and we have put

$$\rho_{\mu\lambda}=\nabla_{\mu}\rho_{\lambda}-\rho_{\mu}\rho_{\lambda}+\frac{1}{2}g_{\mu\lambda}\rho_{\kappa}\rho^{\kappa}. \tag{6}$$

If M and $'M$ are Einstein manifolds,

$$\begin{aligned} K_{\mu\lambda}&=(n-1)kg_{\mu\lambda},\\ 'K_{\mu\lambda}&=(n-1)'k'g_{\mu\lambda}=(n-1)'k\rho^2 g_{\mu\lambda}, \end{aligned} \tag{7}$$

then, from (4) and (5), we have

$$\rho_{\mu\lambda}=\frac{1}{2}(k-'k\rho^2)g_{\mu\lambda} \tag{8}$$

or

$$\nabla_{\mu}\rho_{\lambda}-\rho_{\mu}\rho_{\lambda}=\frac{1}{2}(k-'k\rho^2-\rho_{\kappa}\rho^{\kappa})g_{\mu\lambda}. \tag{9}$$

Hence the conformal transformation is concircular. Q.E.D.

Combining Theorems B and C with this theorem, we can generalize the Yano and Nagano's theorem as follows:

Theorem 2. *If a complete Einstein manifold M is transformed conformally into an Einstein one 'M, then the manifold M is*

I) *a Euclidean space, if* $k=0$,

II) *a spherical space, if* $k>0$, *or*

III) *a hyperbolic space, if* $k<0$.

Theorem 3. *If a complete Einstein manifold M admits a conformal transformation onto itself, then the manifold M is a spherical space.*

References

[1] S. Ishihara and Y. Tashiro: On Riemannian manifolds admitting a concircular transformation, Math. J. Okayama Univ., **9**, 19–47 (1959).

[2] K. Yano: Concircular geometry, I–V, Proc. Imp. Acad., **16**, 195–200, 354–360, 442–448, 505–511 (1940); **18**, 446–451 (1942).

[3] K. Yano and T. Nagano: Einstein spaces admitting a one-parameter group of conformal transformations, Ann. Math., **69**, 451–461 (1959).