53. A Characterization of Holomorphically Complete Spaces

By Ryôsuke IWAHASHI
Mathematical Institute, Nagoya University
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Given a connected complex space X, we denote by $A(X)$ the C-algebra of holomorphic functions on X. A C-homomorphism of $A(X)$ into C which preserves the constants is called a *character* of $A(X)$. Let X^* be the set of all characters of $A(X)$. The functions of $A(X)$ can be considered as functions on X^*. We shall consider X^* as a topological space: the open sets of X^* are those which can be represented as unions of sets of the form $f_1^{-1}(U_1)\frown\cdots\frown f_k^{-1}(U_k)$, where $f_1,\cdots,f_k$ are in $A(X)$, while $U_1,\cdots,U_k$ are open subsets of C ($f^{-1}(U)$ denotes the set of characters χ such that $\chi f\in U$). The space X^* is a Hausdorff space. We assign to each $x\in X$ a point $\theta(x)$ of X^* which is defined by $\theta(x)f=f(x)$ for every $f\in A(X)$. The mapping $\theta: X\to X^*$ is continuous.

Theorem. Let X be a connected complex space. Then X is holomorphically complete if and only if $\theta: X\to X^*$ is a homeomorphism.

For holomorphically complete spaces, see H. Cartan [1] and H. Grauert [2].

Proof. Suppose that X is holomorphically complete. Since X is holomorphically separable [2], the mapping θ is injective. Let χ be a point of X^*. We denote by M the maximal ideal Ker χ. Take $f_1\neq 0$ in M and decompose the analytic set $V^{(1)}=\{x\in X\,|\,f_1(x)=0\}$ of dimension $n-1$ (X being of dimension n) into irreducible components $V_i^{(1)}$. The family $(V_i^{(1)})$ being locally finite, we can find two points x_i, x_i' in $V_i^{(1)}$ for each i such that all the points are distinct and form an analytic set, of dimension 0, in X. By Theorem B on holomorphically complete spaces [1], we can find a function f in $A(X)$ such that $f(x_i)=0$ and $f(x_i')=1$ for every i. Let $f_2=f-\chi f$. Then $f_2\in M$ is not identically zero on each $V_i^{(1)}$. Decompose the analytic set $V^{(2)}=\{x\in X\,|\,f_1(x)=f_2(x)=0\}$ of dimension $n-2$ into irreducible components and find $f_3\in M$ as before. The repetition of such processes leads to the analytic set $V^{(n)}=\{x\in X\,|\, f_1(x)=\cdots=f_n(x)=0\}$ of dimension 0 in X, where $f_1,\cdots,f_n\in M$. Applying Theorem B again, we can find a function $f\in A(X)$ which takes different values at distinct points of $V^{(n)}$. Let $f_{n+1}=f-\chi f$. By Theorem A [1] we know that any finite subset of $A(X)$ without common zero generates $A(X)$ over itself. Therefore the functions $f_1,\cdots,f_{n+1}$ have at least one, and so only one, common zero, say x. For any $f\in M$, then functions $f_1,\cdots,f_{n+1},f$ have the common zero x and so $f(x)=0$, that is, $f\in \text{Ker}\,\theta(x)$.

By the maximality of M we conclude $M=\operatorname{Ker}\theta(x)$, that is, $\chi=\theta(x)$. Thus the mapping θ is surjective.

Let Ω be an ultra-filter on X^* which converges to a point $\chi\in X^*$. If the ultra-filter $\theta^{-1}\Omega$ does not converge in X, we can find a function $f\in A(X)$ such that f tends to infinity along $\theta^{-1}\Omega$, because X is holomorphically convex. Therefore Ωf converges to infinity, while, f being continuous in X^*, Ωf tends to $\chi f\neq\infty$. This contradiction shows that $\theta^{-1}\Omega$ converges to a point $\bar{x}\in X$. Then Ω converges to $\theta(\bar{x})$ by the continuity of θ and we have $\bar{x}=\theta^{-1}\chi$ by the injectivity of θ. Thus the inverse mapping θ^{-1} is continuous.

We have to prove the converse.

Lemma. A complex space is K-complete if it is holomorphically separable.

Proof of Lemma. Let X be a complex space which is holomorphically separable. We see that, for any compact set K in X, there exists a finite number of functions in $A(X)$ which separates the points of K. For any point $x\in X$, let U be a compact neighborhood of x. Take functions $f_1,\cdots,f_k$ which separate the points of U. Denoting by τ the mapping generated by $f_1,\cdots,f_k$ of X into C^k, we have $\tau^{-1}\tau(x)\frown U=\{x\}$. Therefore τ is non-degenerate at x. Thus the space X is K-complete.

Now, suppose that $\theta:X\to X^*$ is a homeomorphism. Since θ is injective, X is holomorphically separable. By Lemma we conclude that X is K-complete.

Suppose that X is not holomorphically convex. Then there exist a compact set K with non-compact envelope $\hat{K}$ of holomorphy and an ultra-filter base $\mathfrak{A}$ on $\hat{K}$ which does not converge in X. For each $f\in A(X)$ the ultra-filter base $f(\mathfrak{A})$ on the closed disk $\{z\in C\mid |z|\leq\sup_K|f|\}$ converges to a uniquely determined point, say χf. We see that the mapping χ of $A(X)$ into C is a character of $A(X)$. Since the mapping θ is surjective, there exists a point x such that $\theta(x)=\chi$. The ultra-filter base $\theta(\mathfrak{A})$ converges to χ, and so $\mathfrak{A}=\theta^{-1}\theta(\mathfrak{A})$ converges to $x=\theta^{-1}\chi$. This is a contradiction. Thus X is holomorphically convex.

References

[1] H. Cartan: Séminaire E. N. S. (1951-1952).
[2] H. Grauert: Charakterisierung der holomorph-vollständigen komplexen Räume, Math. Ann., **129**, 233-259 (1955).