48. On Quasi-normed Spaces. III

By Tomoko KONDA

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In this paper, we consider the inverse of a linear transformation of a (QN) space into a (QN) space. Here, we consider a linear transformation T whose domain is a (QN) space E with the power r $(0<r\leq 1)$ and range is a (QN) space F with the power s $(0<s\leq 1)$, see [2], [3] or [4].

If a linear transformation T is one-to-one, then T has the inverse transformation T^{-1} of F onto E.

Theorem 1. *A linear transformation T has a bounded inverse if and only if there exists a positive number m such that* $\|T(x)\|_s \geq m\|x\|_r^{\frac{s}{r}}$ *for all* $x\in E$.

Proof. Suppose that T has a bounded inverse T^{-1}, then there exists M such that $\|T^{-1}(y)\|_r\leq M\|y\|_s^{\frac{r}{s}}$, and there exists $x\in E$ such that $y=T(x)$. Therefore,

$$\|T^{-1}(T(x))\|_r\leq M\|T(x)\|_s^{\frac{r}{s}},$$

$$\|x\|_r\leq M\|T(x)\|_s^{\frac{r}{s}}$$

and

$$\|x\|_r^{\frac{s}{r}}\leq M^{\frac{s}{r}}\|T(x)\|_s.$$

Let $M^{\frac{s}{r}}=m^{-1}$, then we have $m\|x\|_r^{\frac{s}{r}}\leq\|T(x)\|_s$.

To prove the inverse, let $\|T(x)\|_s=0$, then $T(x)=0$ and $x=0$. On the other hand $x=0$ implies $m\|x\|_r^{\frac{s}{r}}=0$. Therefore T is one-to-one and has the inverse T^{-1} of T.

In Theorem 1, we can take m as the norm $\|T\|_s$ of the transformation, i.e. $\|T(x)\|_s\geq\|T\|_s\|x\|_r^{\frac{s}{r}}$. Consequently, the norm of inverse transformation is defined by $\|T^{-1}\|_r=\|T\|_s^{-\frac{r}{s}}$, hence we have $\|T^{-1}\|_r^s=\|T\|_s^{-r}$.

Now, we shall show that a well-known Banach theorem on inverse transformation is also true for the case of (QN) spaces. First, we shall prove Lemmata.

Lemma 1. *Let T be a bounded linear transformation of E into F. If the image under T of the unit sphere S_1 in E is dense in some sphere U_r about the origin of F, then $T(S_1)$ includes U_r.*

Proof. By the assumption, the set $A=U_r\frown T(S_1)$ is dense in U_r. Let y be any point of U_r. For any $\delta>0$, we take $y_0=0$ and choose inductively a sequence $y_n\in F$ such that $y_{n+1}-y_n\in\delta^n A$ and $\|y_{n+1}-y_n\|$

$<\delta^{(n+1)s}r$ for all $n\geq 0$. Therefore, there exists a sequence x_n such that $T(x_{n+1})=y_{n+1}-y_n$ and $||x_{n+1}||<\delta^{ns}$. If we put $x=\sum_1^\infty x_n$, then

$$||x||_r\leq\sum_1^\infty ||x_n||_r<\sum_0^\infty \delta^{ns}=\frac{1}{1-\delta^s}$$

and

$$T(x)=\sum_1^\infty (y_n-y_{n-1})=y.$$

This implies the image of the sphere of radius $1/1-\delta^s$ covers U_r. For δ is arbitrary, $T(S_1)$ covers U_r.

Lemma 2. *If the image of S_1 under T is dense in no sphere of F, then the range of T includes no sphere of F.*

Proof. Let $T(S_1)$ be not dense in any sphere of F, then $T(S_n)=\{T(x);\ ||x||_s<n\}=n^{\frac{1}{r}}T(S_1)$ is not dense in F. For any sphere $S\subset F$, there exists a closed sphere $S(y_1, r_1)\subset S$ and it is disjoint from $T(S_1)$, and by the induction, exists a sequence of closed spheres $S(y_n, r_n)\subset S(y_{n-1}, r_{n-1})$ such that $S(y_n, r_n)$ is disjoint from $T(S_n)$. Now, we can choose that $r_n\to 0$, and then the sequence $\{y_n\}$ is Cauchy. Its limit y is included in all the spheres $S(y_n, r_n)$ and not included in all the sets $T(S_n)$. By $\bigcup_n T(S_n)=T(X)$, $T(X)$ does not include any sphere in F. The proof is complete. Next, we shall show the following.

Theorem 2. *If T is one-to-one bounded linear transformation of E onto F, then T^{-1} is bounded.* (For the usual case, see [1].)

Proof. By Lemma 2, $T(S_1)$ is dense in some sphere in F, and then $T(S_2)$ is dense in a sphere U_R. But $U_R\subset T(S_2)$ by Lemma 1, $T^{-1}(U_R)\subset S_2$ and $||T^{-1}||_r\leq 2R^{-\frac{r}{s}}$.

Corollary. *Suppose that $r\leq s$ in a (QN) space E with the power r and a (QN) space F with the power s and the graph of a linear transformation T of E into F is closed, then T is bounded.*

Proof. Let $||(x, Tx)||=||x||_r+||Tx||_s^{\frac{r}{s}}$, then it is a quasi-norm with the power r and by the assumption the graph of T is a (QN) space with the power r. By Theorem 2, the transformation $(x, Tx)\to x$ which is norm decreasing and onto E has the inverse transformation and it is bounded. Hence, there exists a constance C such that $||x||_r+||Tx||_s^{\frac{r}{s}}\leq C||x||_r^{\frac{s}{r}}$, $||Tx||_s^{\frac{r}{s}}\leq C||x||_r^{\frac{s}{r}}$ and $||Tx||_s\leq C^{\frac{s}{r}}||x||_r^{\frac{s^2}{r^2}}$. This implies the bound of T.

Corollary is a generalization of the closed graph theorem for a case of (QN) spaces.

References

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