79. On a Metric Characterizing Dimension

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As well known, a separable metric space has dimension$\leqq n$ if and only if it admits a topology-preserving metric such that almost all of the spherical nbds (=neighborhoods) of any point have boundaries of dimension$\leqq n-1$ [1, 2]. To extend this theorem to a non-separable metric space and its covering dimension, we must face two problems. The first of them is how to modify the above condition of metric to fit it for the non-separable case, because in that case, we can not regard this condition as a sufficient condition for n-dimensionality so far as the well-known conjecture, $\dim R=\operatorname{ind}\dim R$, has not yet been solved. The second is how to manage the proof in a non-separable metric space R without a measure, because, although the above theorem was originally proved by virtue of Szpilrajn's theorem on the so-called p-dimensional measure and dimension [3], the measure does not work at all in a general metric space.

After all we can insist the following theorem for a general metric space R and the covering dimension of R.

Theorem. *A metric space R has dimension$\leqq n$ if and only if it admits a topology-preserving metric such that the spherical nbds $S_{\frac{1}{2^i}}(p)$, $i=1,2,\cdots$ of any point p have boundaries of dimension$\leqq n-1$ and such that $\{S_{\frac{1}{2^i}}(p)\mid p\in R\}$ is closure preserving for every i.*

Remark. We denote by $S_\varepsilon(p)$ the spherical nbd of p with a radius ε, i.e. $S_\varepsilon(p)=\{q\mid\rho(p,q)<\varepsilon\}$. We call a collection $\{S_\gamma\mid\gamma\in\Gamma\}$ of subsets '*closure preserving*' if $\bigcup_{\gamma\in\Delta}\bar{S}_\gamma=\overline{\bigcup_{\gamma\in\Delta}S_\gamma}$ for any subset Δ of Γ. The metric in this theorem is a particular one; the metrics of Euclidean spaces, for instance, do not satisfy the second condition. To replace $S_{\frac{1}{2^i}}(p)$, $i=1,2,\cdots$ in this theorem by more spherical nbds will be another interesting problem.

Proof. *Sufficiency*: First, let us note that $\{BS_{\frac{1}{2^i}}(p)\mid p\in A\}$ is closure preserving and $B[\bigcup\{S_{\frac{1}{2^i}}(p)\mid p\in A\}]\subset\bigcup\{BS_{\frac{1}{2^i}}(p)\mid p\in A\}$ for every subset A of R, for $\{S_{\frac{1}{2^i}}(p)\mid p\in R\}$ is closure preserving, where we denote by BS the boundary of S. Hence $\dim\bigcup\{BS_{\frac{1}{2^i}}(p)\mid p\in A\}\leqq n-1$ follows from $\dim BS_{\frac{1}{2^i}}(p)\leqq n-1$, $p\in A$ by virtue of a theorem due to K.

Nagami [4], and accordingly we get $\dim B[\cup\{S_{\frac{1}{2^i}}(p)\mid p\in A\}]\leqq n-1$.

Let F and G be given disjoint closed sets; then there exist, by the above notice, two sequences $U_1\supset\overline{U}_2\supset U_2\supset\cdots\supset F$ and $W_1\supset\overline{W}_2\supset W_2\supset\cdots\supset G$ of open sets U_i and W_i such that

$$F=\bigcap_{i=1}^{\infty}U_i,\quad G=\bigcap_{i=1}^{\infty}W_i,\quad \dim(\overline{U}_i-U_i)\leqq n-1,\quad \dim(\overline{W}_i-W_i)\leqq n-1,\quad i=1,2,\cdots .$$

It is clear that $U=\bigcup_{i=1}^{\infty}(U_i-\overline{W}_i)$ is an open set satisfying $F\subset U\subset R-G$, $\overline{U}-U\subset\bigcup_{i=1}^{\infty}\{(\overline{U}_i-U_i)\cup(\overline{W}_i-W_i)\}$ that implies $\dim(\overline{U}-U)\leqq n-1$ in view of the sum theorem and accordingly $\dim R\leqq n$ can be concluded.

Necessity: Let $\{R\}=\mathfrak{V}_1>\mathfrak{V}_2^{***}>\mathfrak{V}_2>\mathfrak{V}_3^{***}>\cdots$ be a sequence of locally finite open coverings $\mathfrak{V}_i$, $i=1,2,\cdots$ such that $\{S(p,\mathfrak{V}_i)\mid i=1,2,\cdots\}$ is a nbd basis of each point p and such that $\dim B(V)\leqq n-1$ for every $V\in\bigcup_{i=1}^{\infty}\mathfrak{V}_i$ (in this paper we owe notations about covering to [5]). The existence of such a sequence is assured by the n-dimensionality of R.

Now we define coverings $\mathfrak{U}_{\frac{1}{2^i}}$ and $\mathfrak{U}_{\frac{1}{2^i}\left(1-\frac{1}{2^k}\right)}$ for $i=1,2,\cdots$, $k=i+1, i+2,\cdots$ by $\mathfrak{U}_{\frac{1}{2^i}}=\mathfrak{V}_i$, $\mathfrak{U}_{\frac{1}{2^i}\left(1-\frac{1}{2^k}\right)}=\{R-S(R-U,\ \mathfrak{U}_{\frac{1}{2^k}})\mid U\in\mathfrak{U}_{\frac{1}{2^i}}\}$; then $r\leqq r'$ obviously implies $\mathfrak{U}_r<\mathfrak{U}_{r'}$. It is easy to see that $\varphi(x,y)=\inf\{r\mid x\in S(y,\mathfrak{U}_r)\}$ satisfies the following three conditions,

1) $\varphi(x,y)\geqq 0$; $\varphi(x,y)=0$ if and only if $x=y$,
2) $\varphi(x,y)=\varphi(y,x)$,
3) $\varphi(x,y)<\varepsilon$ and $\varphi(y,z)<\varepsilon$ for a positive number ε imply $\varphi(x,z)<2\varepsilon$.

To check the third condition, we assume, for instance, $\varphi(x,y)\leqq\varphi(y,z)\neq 0$. Since the value of $\varphi(y,z)$ must be either $\frac{1}{2^i}$ or $\frac{1}{2^i}\left(1-\frac{1}{2^k}\right)$ for some i and k, let $\varphi(y,z)=\frac{1}{2^i}\left(1-\frac{1}{2^k}\right)<\varepsilon$; then there exist $U_1, U_2\in\mathfrak{U}_{\frac{1}{2^i}}$ such that $x,y\in R-S(R-U_1,\mathfrak{U}_{\frac{1}{2^k}})$ and $y,z\in R-S(R-U_2,\mathfrak{U}_{\frac{1}{2^k}})$. Since $y\in U_1\frown U_2\neq\phi$, there must be $U_3\in\mathfrak{U}_{\frac{1}{2^{i-1}}}(>\mathfrak{U}^{*}_{\frac{1}{2^i}})$ with $U_3\supset U_1\cup U_2$. Hence $x,z\in R-S(R-U_3,\mathfrak{U}_{\frac{1}{2^k}})\in\mathfrak{U}_{\frac{1}{2^{i-1}}\left(1-\frac{1}{2^k}\right)}$, which implies $\varphi(x,z)\leqq\frac{1}{2^{i-1}}\left(1-\frac{1}{2^k}\right)<2\varepsilon$. If the value of $\varphi(y,z)$ would have a form of $\frac{1}{2^i}$, the proof would be much easier. At any rate, the point is in the fact that $\mathfrak{U}^{*}_r<\mathfrak{U}_{2r}$. Thus we get, by A. H. Frink's lemma [6], a metric function $\rho(p,q)$ of R that is defined by

$$\rho(p,q)=\inf\{\varphi(p,x_1)+\varphi(x_1,x_2)+\cdots+\varphi(x_m,q)\mid x_j\in R,\ j=1,\cdots,m;\ m=1,2,\cdots\}$$

and satisfies
$$\frac{1}{4}\varphi(p,q)\leqq\rho(p,q)\leqq\varphi(p,q).$$

Now, the problem is to show that ρ is the desired metric. The principal point of the proof is to show $S(p, \mathfrak{U}_{\frac{1}{2^i}}) = S_{\frac{1}{2^i}}(p)$ for every $p \in R$.

A) First, it is clear that $q \in S(p, \mathfrak{U}_{\frac{1}{2^i}})$ implies $\rho(p, q) < \frac{1}{2^i}$, for $p, q \in U \in \mathfrak{U}_{\frac{1}{2^i}}$ implies $p, q \in R - S(R-U, \mathfrak{U}_{\frac{1}{2^k}})$ for a sufficiently large k, which means $\rho(p, q) \leqq \frac{1}{2^i}\left(1 - \frac{1}{2^k}\right) < \frac{1}{2^i}$.

B) Next, we shall show that $q \notin S(p, \mathfrak{U}_{\frac{1}{2^i}})$ implies $\rho(p, q) \geqq \frac{1}{2^i}$. To show this, it suffices, in view of the definition of ρ, to show that if

$$U_j \frown U_{j+1} \neq \phi,\ j=1,\cdots,m-1\ (m \geqq 2);\ U_j \in \mathfrak{U}_{r_j},\ j=1,\cdots,m;$$
$$p \in U_1,\ q \in U_m, \quad \text{then} \quad \sum_{j=1}^{m} r_j \geqq \frac{1}{2^i}.$$

1) In case of $r_j \leqq \frac{1}{2^{i+1}}$, $j=1,\cdots,m$, we define new coverings $\mathfrak{U}'_{\frac{1}{2^h}}$ and $\mathfrak{U}'_{\frac{1}{2^h}(1-\frac{1}{2^k})}$ by

$$\mathfrak{U}'_{\frac{1}{2^h}} = \mathfrak{U}_{\frac{1}{2^h}},\quad \mathfrak{U}'_{\frac{1}{2^h}(1-\frac{1}{2^k})} = \mathfrak{U}_{\frac{1}{2^h}(1-\frac{1}{2^k})}, \qquad h=i+1, i+2, \cdots,\ k=h+1, h+2, \cdots,$$
$$\mathfrak{U}'_{\frac{1}{2^i}} = \mathfrak{U}'^{*}_{\frac{1}{2^{i+1}}},\quad \mathfrak{U}'_{\frac{1}{2^i}(1-\frac{1}{2^k})} = \{R - S(R-U, \mathfrak{U}_{\frac{1}{2^k}}) \mid U \in \mathfrak{U}'_{\frac{1}{2^i}}\}, \qquad k=i+1,\ i+2, \cdots,$$
$$\mathfrak{U}'_{\frac{1}{2^{i-1}}} = \mathfrak{U}'^{*}_{\frac{1}{2^i}},\quad \mathfrak{U}'_{\frac{1}{2^{i-1}}(1-\frac{1}{2^k})} = \mathfrak{U}'^{*}_{\frac{1}{2^i}(1-\frac{1}{2^k})}, \qquad k=i+1,\ i+2, \cdots,$$
$$\mathfrak{U}'_{\frac{1}{2^{i-2}}} = \mathfrak{U}'^{*}_{\frac{1}{2^{i-1}}},\quad \mathfrak{U}'_{\frac{1}{2^{i-2}}(1-\frac{1}{2^k})} = \mathfrak{U}'^{*}_{\frac{1}{2^{i-1}}(1-\frac{1}{2^k})}, \qquad k=i+1,\ i+2, \cdots,$$
$$\cdots\cdots\cdots\cdots$$
$$\mathfrak{U}'_{\frac{1}{2}} = \mathfrak{U}'^{*}_{\frac{1}{2^2}},\quad \mathfrak{U}'_{\frac{1}{2}(1-\frac{1}{2^k})} = \mathfrak{U}'^{*}_{\frac{1}{2^2}(1-\frac{1}{2^k})}, \qquad k=i+1,\ i+2, \cdots,$$
$$\mathfrak{U}'_1 = \{R\},\quad \mathfrak{U}'_{(1-\frac{1}{2^k})} = \{R\}, \qquad k=i+1,\ i+2, \cdots .$$

Then it is easy to see that $\mathfrak{U}'_r < \mathfrak{U}'_{r'}$ if $r \leqq r'$ and that $\mathfrak{U}'^{*}_r < \mathfrak{U}'_{2r}$ if $r \leqq \frac{1}{2}$. Hence $\varphi'(x, y) = \inf\{r \mid x \in S(y, \mathfrak{U}'_r)\}$ satisfies the previous conditions 1), 2), 3) in the same way as shown there. Therefore $\rho'(p, q) = \inf\{\varphi'(p, x_1) + \cdots + \varphi'(x_m, q) \mid x_j \in R,\ j=1,\cdots,m;\ m=1,2,\cdots\}$ is another metric function of R satisfying $\frac{1}{4}\varphi' \leqq \rho' \leqq \varphi'$. Now, since $U_j \in \mathfrak{U}_{r_j}$ and $r_j \leqq \frac{1}{2^{i+1}}$, $j=1,\cdots,m$, we may regard U_j as a member of $\mathfrak{U}'_{r_j}$ that is equal to $\mathfrak{U}_{r_j}$ by the definition. Hence we get $\frac{1}{4}\varphi'(p, q) \leqq \rho'(p, q) \leqq \sum_{j=1}^{m} r_j$. On the other hand we get

$$S(p, \mathfrak{U}'_{\frac{2^3}{2^{i+1}}}) = S(p, \mathfrak{U}'^{***}_{\frac{1}{2^{i+1}}}) \subset S(p, \mathfrak{U}_{\frac{1}{2^i}}) \not\ni q$$

from the definition of $\mathfrak{U}'_{\frac{2^3}{2^{i+1}}}$ and the fact that $\mathfrak{U}'^{***}_{\frac{1}{2^{i+1}}} = \mathfrak{U}^{***}_{\frac{1}{2^{i+1}}} = \mathfrak{V}^{***}_{i+1} < \mathfrak{V}_i = \mathfrak{U}_{\frac{1}{2^i}}$. This implies $\varphi'(p, q) \geqq \frac{2^3}{2^{i+1}}$, and hence $\sum_{j=1}^{m} r_j \geqq \frac{1}{4} \cdot \frac{2^3}{2^{i+1}} = \frac{1}{2^i}$.

Incidentally, let us note that we may conclude $\sum_{j=1}^{m} r_j > \frac{1}{2^{i+1}}$ whenever

$q \notin S(p, \mathfrak{U}_{\frac{1}{2^i}})$; this remark will be used later.

2) In case of $r_j > \frac{1}{2^{i+1}}$ for some j, we may assume $\frac{1}{2^i} > r_j > \frac{1}{2^{i+1}}$, because $\sum_{j=1}^{m} r_j \geqq \frac{1}{2^i}$ is clear if $r_j \geqq \frac{1}{2^i}$. Hence we can assume $r_j = \frac{1}{2^i}\left(1 - \frac{1}{2^k}\right)$ for some $k \geqq i+1$. Let $R - S(R-U, \mathfrak{U}_{\frac{1}{2^k}}) = U_j \in \mathfrak{U}_{r_j}$; then since $q \notin S(p, \mathfrak{U}_{\frac{1}{2^i}})$, either p or q is not contained in U. Let us, for example, assume $q \notin U$, and let $x \in U_j \frown U_{j+1}$; then since $q \notin S(x, \mathfrak{U}_{\frac{1}{2^k}})$, we get $r_{j+1} + r_{j+2} + \cdots + r_m > \frac{1}{2^{k+1}}$ by the remark at the end of the proof 1). Therefore $\sum_{j=1}^{m} r_j \geqq r_j + r_{j+1} + \cdots + r_m > \frac{1}{2^i}\left(1 - \frac{1}{2^k}\right) + \frac{1}{2^{k+1}} \geqq \frac{1}{2^i}$. Thus we can conclude the validity of B) as well as that of A), which implies $S_{\frac{1}{2^i}}(p) = S(p, \mathfrak{B}_i)$ because of $\mathfrak{B}_i = \mathfrak{U}_{\frac{1}{2^i}}$, and accordingly we get

$$BS_{\frac{1}{2^i}}(p) = BS(p, \mathfrak{B}_i) \subset \smile\{B(V) \mid p \in V \in \mathfrak{B}_i\}.$$

Since $\dim B(V) \leqq n-1$ for every $V \in \mathfrak{B}_i$, we have $\dim BS_{\frac{1}{2^i}}(p) \leqq n-1$ in view of the sum theorem.

Now the only problem is to show that $\{S_{\frac{1}{2^i}}(p) \mid p \in R\}$ is closure preserving. To show it, we deal with a given point q with $q \notin \smile\{\bar{S}_{\frac{1}{2^i}}(p) \mid p \in A\}$ for a subset A of R. Let $\{V \mid V \in \mathfrak{B}_i,\ V \frown A \neq \phi\} = \mathfrak{B}$ and let $U(q)$ be a nbd of q intersecting only finitely many sets V_j, $j = 1, \cdots, m$, in $\mathfrak{B}$. If $q \in \bar{V}_j$, then $q \in \smile\{\bar{S}_{\frac{1}{2^i}}(p) \mid p \in A\}$ contradicting the assumption; hence it must be $q \notin \bar{V}_j$, $j = 1, \cdots, m$. Therefore $W(q) = \bigcap_{j=1}^{m}(R - \bar{V}_j) \frown U(q)$ is a nbd of q satisfying $W(q) \frown S(p, \mathfrak{B}_i) = W(q) \frown S_{\frac{1}{2^i}}(p) = \phi$ for every $p \in A$, which implies the closure preservation of $\{S_{\frac{1}{2^i}}(p) \mid p \in R\}$. Thus the proof of this theorem is complete.

Corollary 1. *A metric space R has dimension $\leqq n$ if and only if it admits a topology-preserving metric such that the spherical nbds $S_{\frac{1}{i}}(p)$, $i = 1, 2, \cdots$ of any point p have boundaries of dimension $\leqq n-1$ and such that $\{S_{\frac{1}{i}}(p) \mid p \in R\}$ is closure preserving for every i.*

Proof. Letting $d(x, x) = 0$, $d(x, y) = -\frac{1}{\log_2 \rho(x, y)}$ $(x \neq y)$ for the metric $\rho(x, y)$ in the theorem, we get a metric $d(x, y)$ satisfying the condition in this corollary.

Corollary 2. *A metric space R has dimension $\leqq n$ if and only if it admits a topology-preserving metric such that* $\dim B[\smile\{S_{\frac{1}{2^i}}(p) \mid p \in A\}] \leqq n-1$ (*or* $\dim B[\smile\{S_{\frac{1}{i}}(p) \mid p \in A\}] \leqq n-1$), $i = 1, 2, \cdots$ *for every subset A of R.*

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