75. A Note on the Milnor's Invariant λ' for a Homotopy 3-sphere

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1. Let M be a differentiable $(4k-1)$-manifold which is a homology sphere and the boundary of some parallelizable manifold W. (The word "manifold" will mean a "compact" manifold throughout in this note.) The intersection number of two homology classes α, β of W will be denoted by $\langle\alpha, \beta\rangle$. Let $I(W)$ be the index of the quadratic form

$$\alpha \to \langle\alpha, \alpha\rangle,$$

where α varies over the Betti group $H_{2k}(W)/(\text{torsion})$. Integer coefficients are to be understood.

Define I_k as the greatest common divisor of $I(M)$ where M ranges over all almost parallelizable manifolds[1)] without boundary of dimension $4k$. The residue class $\frac{1}{8}I(W)$[2)] modulo $\frac{1}{8}I_k$ will be denoted by $\lambda'(M)$.

Then J. Milnor [1] showed the followings:

(1) $\lambda'(M)$ depends only on the J-equivalence[3)] class of M,

(2) λ' gives rise to an isomorphism onto

$$\Lambda' : \Theta^{4k-1}(\partial\pi) \to Z_{\frac{1}{8}I_k} \quad \text{provided that } k>1,$$

where $\Theta^{4k-1}(\partial\pi)$[4)] is the set of all J-equivalence classes of homotopy $(4k-1)$-spheres which are the boundaries of some parallelizable manifolds.

Finally, in the list of unsolved problems (see [1]), he proposed the following:

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1) A manifold M will be called almost parallelizable if there exists a finite subset F so that $M-F$ is parallelizable.

2) The index $I(W)$ of an almost parallelizable manifold is always divisible by 8, provided that ∂W is a homology sphere (see J. Milnor [1]).

3) Two unbounded manifolds M_1, M_2 of the same dimension are J-equivalent if there exists a manifold W such that

(1) the boundary ∂W is the disjoint union of M_1 and $-M_2$,

and

(2) both M_1 and M_2 are deformation retracts of W.

4) $\Theta^{4k-1}(\partial\pi)$ forms an abelian group under the sum operation $\#$, where $\#$ means the following. Let M_1, M_2 be connected differentiable (or combinatorial) manifolds of the same dimension n. The differentiable (or combinatorial) sum $M_1 \# M_2$ is obtained by removing a differentiable (or a combinatorial) n-cell from each, and then pasting properly the resulting boundary together (see J. Milnor [1, §2] and H. Seifert-W. Threlfall [7, Problem 3, p. 218]).

Problem. Does there exist a homotopy 3-sphere M such that $\lambda'(M)\neq 0$?

In this note we shall give a negative answer to this question, that is,

Theorem 1. For any differentiable 3-manifold M which is a homotopy sphere, $\lambda'(M)=0$.

In the course of the proof of this theorem, the following is also proved.

Theorem 2. If any simply connected closed 3-manifold embedded semi-linearly in the 4-sphere is homeomorphic to the 3-sphere S^3, then any simply connected closed 3-manifold is homeomorphic to S^3.

2. Outline of the proofs. Let M be a topological 3-manifold which is a homotopy 3-sphere and let Δ be a 3-simplex of some fixed triangulation[5] of M. After V. Poénaru [6], $(M-\text{Int }\Delta)\times I^2$ is semi-linearly homeomorphic to I^5, where I^n means the n-cube $(x_1,\cdots,x_n)$, $0\leqq x_i\leqq 1$. As the boundary of $(M-\text{Int }\Delta)\times I^2$ is semi-linearly homeomorphic to the boundary of I^5, we may embed $(M-\text{Int }\Delta)$ semi-linearly in the 4-sphere S^4. Let U be the neighborhood[6] of $(M-\text{Int }\Delta)$ in S^4. Then the boundary N of U has the following properties:

(1) N is a simply connected closed 3-manifold embedded semi-linearly in S^4,

(2) N is homeomorphic to $M\# M$, where $M\# M$ means the combinatorial sum[4] of M and M. From the above fact, if any simply connected closed 3-manifold embedded semi-linearly in S^4 is homeomorphic to S^3, $M\# M$ must be homeomorphic to S^3. After E. E. Moise [2], it may therefore be concluded that M is itself homeomorphic to S^3. This proves Theorem 2.

Now we proceed to the proof of Theorem 1. Let M have a differentiable structure. As N is a 3-manifold which is semi-linearly embedded in S^4, we may construct (see [5]) a differentiable 3-manifold V which is differentiablly embedded in S^4 and is homeomorphic to N. As any 3-manifold has a uniquely determined differentiable structure by J. Munkres, S. Smale and J. H. C. Whitehead [4], V is diffeomorphic to the differentiable sum[4] $M\# M$ of M and M. V divides S^4 into two parts, one of which we denote by W. Then W is a parallelizable 4-manifold which is differentiablly embedded in S^4 and has the boundary V. As V is simply connected, we obtain $H_2(W)=0$ by the Alexander's duality theorem. Thus we obtain

$$\tfrac{1}{8}I(W)=\lambda'(V)=\lambda'(M\# M)=2\lambda'(M)=0.$$

This proves Theorem 1.

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5) After E. E. Moise [3], any 3-manifold has a triangulation.

6) See J. H. C. Whitehead [8, p. 290].

References

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