100. General Crossnorms

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(Comm. by K. KUNUGI, M.J.A., July 12, 1960)

In the present note we shall generalize the construction of "general" crossnorm given by R. Schatten in Appendix II in his book: A theory of cross-spaces, Ann. of Math. Studies, no. 26, Princeton (1950) to which we shall refer as [TCS] in this note. And we shall prove some properties of our general crossnorms.

Our construction of general crossnorms is a slight modification of R. Schatten's method. Consequently our proofs of Lemmas 1, 2, 3 and Theorem 1 are almost analogous to those of his Theorems, but for the benefit of readers we shall prove them. Moreover we shall make use of notations, terminologies and results shown in [TCS] without reservation.

Throughout the present note we shall assume that $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ represent perfectly general Banach spaces while $\boldsymbol{B}_1^*$ and $\boldsymbol{B}_2^*$ stand for their conjugate spaces respectively.

LEMMA 1. *If p, q are positive and $\frac{1}{p}+\frac{1}{q}=1$, then for positive numbers a, b, we have*

$$\frac{1}{\frac{a}{p}+\frac{b}{q}} \leqq \frac{1}{pa}+\frac{1}{qb}.$$

Proof. Immediately.

LEMMA 2. *If p, q are positive and $\frac{1}{p}+\frac{1}{q}=1$, then for any two norms α, β, we have*

$$\left(\frac{\alpha}{p}+\frac{\beta}{q}\right)' \leqq \frac{\alpha'}{p}+\frac{\beta'}{q}.$$

Proof. Let $\widetilde{F} \in \boldsymbol{B}_1^* \odot \boldsymbol{B}_2^*$ be fixed. By Lemma 1, for any non-zero $\widetilde{f} \in \boldsymbol{B}_1 \odot \boldsymbol{B}_2$ we have

$$\frac{|\widetilde{F}(\widetilde{f})|}{\frac{\alpha(\widetilde{f})}{p}+\frac{\beta(\widetilde{f})}{q}} \leqq \frac{\widetilde{F}(\widetilde{f})}{p\alpha(\widetilde{f})}+\frac{\widetilde{F}(\widetilde{f})}{q\beta(\widetilde{f})}.$$

Thus, definition of "associated" norm furnishes the proof.

We proceed with our construction:

Put $\alpha_{p,1}=\frac{\gamma}{p}+\frac{\gamma'}{q}$, $\alpha_{q,1}=\frac{\gamma}{q}+\frac{\gamma'}{p}$ and $\alpha_{p,n}=\frac{\alpha_{p,n-1}}{p}+\frac{\alpha'_{q,n-1}}{q}$,

$\alpha_{q,n}=\frac{\alpha_{q,n-1}}{q}+\frac{\alpha'_{p,n-1}}{p}$ for $n=2,3,\cdots$, where γ is the greatest crossnorm which is uniquely defined on $\boldsymbol{B}_1 \odot \boldsymbol{B}_2$ and p, q are positive and $\frac{1}{p}+\frac{1}{q}=1$. Then by Lemma 2 we have

$$\alpha'_{p,n}=\left(\frac{\alpha_{p,n-1}}{p}+\frac{\alpha'_{q,n-1}}{q}\right)' \leqq \frac{\alpha'_{p,n-1}}{p}+\frac{\alpha''_{q,n-1}}{q} \leqq \frac{\alpha'_{p,n-1}}{p}+\frac{\alpha_{q,n-1}}{q}=\alpha_{q,n}$$

therefore $\alpha'_{p,n} \leqq \alpha_{q,n}$. Similarly we have $\alpha'_{q,n} \leqq \alpha_{p,n}$. Then

$$\alpha_{p,n}=\frac{\alpha_{p,n-1}}{p}+\frac{\alpha'_{q,n-1}}{q} \leqq \frac{\alpha_{p,n-1}}{p}+\frac{\alpha_{p,n-1}}{q}=\alpha_{p,n-1},$$

therefore, $\alpha_{p,n} \leqq \alpha_{p,n-1}$, $\alpha_{q,n} \leqq \alpha_{q,n-1}$, $\alpha'_{p,n} \geqq \alpha'_{p,n-1}$ and $\alpha'_{q,n} \geqq \alpha'_{q,n-1}$. Thus we have the following Lemma:

LEMMA 3.

$$\gamma \geqq \alpha_{p,1} \geqq \alpha_{p,2} \geqq \cdots \geqq \alpha_{p,n} \geqq \cdots \geqq \alpha'_{q,n} \geqq \cdots \geqq \alpha'_{q,2} \geqq \alpha'_{q,1} \geqq \gamma',$$
$$\gamma \geqq \alpha_{q,1} \geqq \alpha_{q,2} \geqq \cdots \geqq \alpha_{q,n} \geqq \cdots \geqq \alpha'_{p,n} \geqq \cdots \geqq \alpha'_{p,2} \geqq \alpha'_{p,1} \geqq \gamma'.$$

Then we get

THEOREM 1. *Put* $\lim_{n\to\infty}\alpha_{p,n}=\alpha_p$ *and* $\lim_{n\to\infty}\alpha_{q,n}=\alpha_q$, *then we have* $\alpha'_p=\alpha_q$ *and* $\alpha_p=\alpha''_p=\alpha'_q$ (*therefore, crossnorms* α_p, α_q *are reflexive*).

Proof. We have

$$\alpha_{p,1}-\alpha'_{q,1} \leqq \frac{\gamma}{p}+\frac{\gamma'}{q}-\gamma'=\frac{\gamma}{p}-\frac{\gamma'}{p}=\frac{1}{p}(\gamma-\gamma'),$$

$$\alpha_{p,2}-\alpha'_{q,2} \leqq \frac{\alpha_{p,1}}{p}+\frac{\alpha'_{q,1}}{q}-\alpha'_{q,1}=\frac{1}{p}(\alpha_{p,1}-\alpha'_{q,1}) \leqq \frac{1}{p^2}(\gamma-\gamma'),$$

and in general, $\alpha_{p,n}-\alpha'_{q,n} \leqq \frac{1}{p^n}(\gamma-\gamma')$. Similarly we have $\alpha_{q,n}-\alpha'_{p,n} \leqq \frac{1}{q^n}(\gamma-\gamma')$. Put $\lim_{n\to\infty}\alpha'_{q,n}=\beta_1$ and $\lim_{n\to\infty}\alpha'_{p,n}=\beta_2$. Since $p>1$ and $q>1$, we have $\alpha_p=\beta_1$ and $\alpha_q=\beta_2$. Since $\alpha_{p,n} \geqq \alpha_p$ and consequently $\alpha'_{p,n} \leqq \alpha'_p$ for all n, we have $\alpha'_p \geqq \beta_2=\alpha_q$. Similarly, $\alpha'_q \geqq \alpha_p$. On the other hand, since $\alpha_q=\beta_2 \geqq \alpha'_{p,n}$ and hence $\alpha'_q \leqq \alpha''_{p,n} \leqq \alpha_{p,n}$ for all n, we have $\alpha'_q \leqq \alpha_p$. Thus we have $\alpha_p=\alpha'_q$. Therefore similarly we have $\alpha'_p=\alpha_q=\alpha''_q$.

LEMMA 4. *If we assume that* $p>q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then we have* $\alpha_p \leqq \alpha_q$.

Proof. Since

$$\alpha_{q,1}-\alpha_{p,1}=\left(\frac{\gamma}{q}+\frac{\gamma'}{p}\right)-\left(\frac{\gamma}{p}+\frac{\gamma'}{q}\right)=\left(\frac{1}{q}-\frac{1}{p}\right)(\gamma-\gamma') \geqq 0,$$

we have $\alpha_{q,1} \geqq \alpha_{p,1}$. We shall apply mathematical induction for n. If we assume that $\alpha_{q,n} \geqq \alpha_{p,n}$ and consequently $\alpha'_{q,n} \leqq \alpha'_{p,n}$, then we have

$$\begin{aligned}\alpha_{q,n+1}-\alpha_{p,n+1}&=\left(\frac{\alpha_{q,n}}{q}+\frac{\alpha'_{p,n}}{p}\right)-\left(\frac{\alpha_{p,n}}{p}+\frac{\alpha'_{q,n}}{q}\right)\\ &\geqq\left(\frac{\alpha_{q,n}}{q}+\frac{\alpha'_{q,n}}{p}\right)-\left(\frac{\alpha_{q,n}}{p}+\frac{\alpha'_{q,n}}{q}\right)\\ &=\left(\frac{1}{q}-\frac{1}{p}\right)(\alpha_{q,n}-\alpha'_{q,n})\\ &\geqq\left(\frac{1}{q}-\frac{1}{p}\right)(\alpha_{q,n}-\alpha_{p,n}) \geqq 0.\end{aligned}$$

Therefore we have $\alpha_{q,n} \geqq \alpha_{p,n}$ for all n. Thus we get $\alpha_q \geqq \alpha_p$.

In more general we have

THEOREM 2. *If $p>r>1$, then we have $\alpha_r \geqq \alpha_p$.*

Proof. If $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{r}+\frac{1}{s}=1$, then $s>q>1$ and $\frac{1}{r}-\frac{1}{p}=\frac{1}{q}-\frac{1}{s}$. Now since

$$\begin{aligned}\alpha_{r,1}-\alpha_{p,1}&=\left(\frac{\gamma}{r}+\frac{\gamma'}{s}\right)-\left(\frac{\gamma}{p}+\frac{\gamma'}{q}\right)\\&=\gamma\left(\frac{1}{r}-\frac{1}{p}\right)-\gamma'\left(\frac{1}{q}-\frac{1}{s}\right)\\&=\left(\frac{1}{r}-\frac{1}{p}\right)(\gamma-\gamma')\geqq 0,\end{aligned}$$

we have $\alpha_{r,1} \geqq \alpha_{p,1}$ and similarly $\alpha_{s,1} \leqq \alpha_{q,1}$. Assume that $\alpha_{r,k} \geqq \alpha_{p,k}$, $\alpha_{s,k} \leqq \alpha_{q,k}$ and consequently $\alpha'_{r,k} \leqq \alpha'_{p,k}$, $\alpha'_{s,k} \geqq \alpha'_{q,k}$, then we have

$$\begin{aligned}\alpha_{r,k+1}-\alpha_{p,k+1}&=\left(\frac{\alpha_{r,k}}{r}+\frac{\alpha'_{s,k}}{s}\right)-\left(\frac{\alpha_{p,k}}{p}+\frac{\alpha'_{q,k}}{q}\right)\\&\geqq\left(\frac{\alpha_{r,k}}{r}+\frac{\alpha'_{q,k}}{s}\right)-\left(\frac{\alpha_{r,k}}{p}+\frac{\alpha'_{q,k}}{q}\right)\\&=\alpha_{r,k}\left(\frac{1}{r}-\frac{1}{p}\right)-\alpha'_{q,k}\left(\frac{1}{q}-\frac{1}{s}\right)\\&=\left(\frac{1}{r}-\frac{1}{p}\right)(\alpha_{r,k}-\alpha'_{q,k})\\&\geqq\left(\frac{1}{r}-\frac{1}{p}\right)(\alpha_{r,k}-\alpha'_{s,k})\geqq 0.\end{aligned}$$

Therefore we have $\alpha_{r,k+1} \geqq \alpha_{p,k+1}$ and similarly $\alpha_{s,k+1} \leqq \alpha_{q,k+1}$. Thus, we have $\alpha_{r,n} \geqq \alpha_{p,n}$ for all n. Therefore we get $\alpha_r \geqq \alpha_p$.

REMARK. We shall discuss elsewhere what position our crossnorms take among unitarily invariant crossnorms in the special case where $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are Hilbert spaces $\boldsymbol{H}$ and $\bar{\bar{\boldsymbol{H}}}$ respectively.