92. Comparability between Ramified Sets

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Introduction. Let X be a partially ordered set and a be its element. Ub(a) and Lb(a) denote the sets $\{x|x\varepsilon X, x>a\}$ and $\{x|x\varepsilon X, x<a\}$ respectively.

DEFINITION 1. i) *A partially ordered set X is called a ramified set if* Lb(a) *is a well-ordered subset of X for any $a\varepsilon X$.* ii) *A non-void ramified set X is called perfectly irresoluble if* Ub(a) *is not totally ordered for any $a\varepsilon X$. X is called irresoluble if it includes a non-void perfectly irresoluble subset. X is called resoluble if any its non-void subset is not perfectly irresoluble.**)

In connection with Souslin's Problem (see [2]), investigations of ramified sets have been proceeded by many authors including Prof. George Kurepa (see [3]). In this paper we are interested in the internal structure of ramified sets and comparison between them, and obtained several results mentioned later, which seem fundamental in the theory of structures of them. But here we only give the outline of their proofs and details will be published elsewhere.

First the following follows from Definition 1.

THEOREM 1. *A ramified set X includes the largest* (*in the sense of inclusion*) *perfectly irresoluble subset $K(X)$. X is resoluble if and only if $K(X)$ is void.*

1. Main Theorems. Hereafter X, Y and Z denote ramified sets.

DEFINITION 2. i) *We write $X\propto Y$ if there exists a mapping f* (*many-to-one in general*) *of X into Y such that $a<x$ implies $f(a)<f(x)$, $X\sim Y$ if $X\propto Y$ and $Y\propto X$, and $X\precnsim Y$ if $X\propto Y$ and $Y\not\propto X$ where $Y\not\propto X$ is the negation of $Y\propto X$.* ii) *Let β be a regular ordinal number greater than 0. $\mathfrak{R}_\beta$ denotes the family of all ramified sets X with $\overline{\overline{X}}<\aleph_\beta$, and $\mathfrak{S}_\beta$ denotes the family of all resoluble sets in $\mathfrak{R}_\beta$.*

$\propto$ is a quasi-ordering and $\sim$ is an equivalence relation between ramified sets. If we identify equivalent sets, $\propto$ becomes an order relation. We shall say that X and Y are *comparable* with each other, if either $X\propto Y$ or $Y\propto X$ holds. Our Main Theorems are the following.

MAIN THEOREM A. i) *$\mathfrak{S}_\beta$ is well-ordered by $\propto$ under identifica-*

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*) Of course a void set is regarded as a well-ordered set (and hence a totally ordered set). For convenience, a void set is regarded as a ramified set which is resoluble and (perfectly) irresoluble in the same time.

tion of equivalent sets. ii) *If one of X and Y is resoluble, then they are mutually comparable.*

MAIN THEOREM B. *Continuum Hypothesis implies that* i), *there exists a pair of ramified sets X and Y such that neither $X \propto Y$ nor $Y \propto X$ holds, and* ii), *there exists a sequence $\{X_n \mid n<\omega\}$ such that $X_{n+1} \not\approx X_n$ for any $n<\omega$.*

MAIN THEOREM C. *$\mathfrak{R}_1$ is well-ordered by $\propto$ under identification of equivalent sets.*

Now we shall mention the outline of proofs of these Theorems.

2. Preliminaries on ordinal numbers. In this paper every small Greek letter stands for an ordinal number, and the letter n denotes a finite number. β is a regular number greater than 0 as we stated in Definition 2. Concerning terminologies and notations mentioned without definitions, refer to [1]. Every number λ is decomposed into a sum $\mu+n$ of a limit number μ and a finite number n, where μ will be denoted by $\mathrm{ls}(\lambda)$ and n will be denoted by $\mathrm{fr}(\lambda)$. Let ν be a number with $0<\nu<\omega_\beta$. If ν is an isolated number, then Φ_ν^i denotes the class of all numbers λ with $\mathrm{fr}(\lambda)>0$ and Φ_ν^l denotes the class of all numbers λ with $\mathrm{fr}(\lambda)=0$. If ν is a limit number, then Φ_ν^i denotes the class of all numbers λ such that there exists a rest ζ of λ with $0<\zeta<\nu$ and Φ_ν^l denotes the class of all numbers λ such that any non-zero rest of λ is not less than ν. Put $\rho_\mu=1$ if $\mathrm{fr}(\mu)=0$ and $\rho_\mu=0$ if $\mathrm{fr}(\mu)>0$. Functions $\alpha_\nu(\lambda)$ of λ, assigned to each ν with $0<\nu<\omega_\beta$, are defined as follows:

DEFINITION 3. *$\alpha_1(\lambda)=\lambda$ and $\alpha_\nu(0)=0$ for any ν. Assume that ν and λ are given and $\alpha_\eta(\mu)$ is defined for any η and μ with $0<\eta<\nu$ and $\alpha_\nu(\mu)$ is defined for any $\mu<\lambda$.*

If $\nu=\eta+1$ and $\lambda=\mu+1$, then put $\alpha_\nu(\lambda)=\alpha_\eta(\omega_\beta^{\alpha_\nu(\mu)+\rho_\mu})$.

If ν is a limit number and $\lambda=\nu\delta+\zeta$ where $0<\zeta<\nu$ (i.e. $\lambda \in \Phi_\nu^i$) *then put $\alpha_\nu(\lambda)=\alpha_\zeta(\omega_\beta^{\alpha_\nu(\nu\delta)+1})$.*

If $0<\lambda$ and $\lambda \in \Phi_\nu^l$, then put $\alpha_\nu(\lambda)=\sup_{\mu<\lambda} \alpha_\nu(\mu)$.

Finally put $\beta^\# = \sup_{\nu<\omega_\beta} \alpha_\nu(1)$.

Concerning the functions $\alpha_\nu(\lambda)$ we have

THEOREM 2. i) *If $0<\eta<\nu<\omega_\beta$, then for any $\mu>0$, except where* $\mathrm{fr}(\nu)=0$ *and $\mu=\nu\delta+\zeta$ with $0<\zeta<\eta$, there exists a λ such that $\alpha_\nu(\mu)=\alpha_\eta(\omega_\beta^\lambda)$.* ii) *$\mu<\lambda<\beta^\#$ implies $\alpha_\nu(\mu)<\alpha_\nu(\lambda)$ for any ν.* iii) *In order that $\lambda=\alpha_\eta(\omega_\beta^\lambda)$ for any $\eta<\nu$, it is necessary and sufficient that $\lambda=\alpha_\nu(\mu)$ with $\mu \in \Phi_\nu^l$.* iv) *$\lambda \leq \alpha_\nu(\lambda)$ for any λ and ν with $0<\nu<\omega_\beta$.* v) *If $\lambda<\beta^\#$, then there exists a $\nu<\omega_\beta$ such that $\lambda<\alpha_\nu(\omega_\beta^\lambda)$.* vi) *If $\nu>1$, then $\alpha_\nu(\lambda)$ is a limit number for any $\lambda<\beta^\#$, and* $\mathrm{cf}(\alpha_\nu(\lambda))=\beta$ *for any $\lambda \in \Phi_\nu^i$ and* $\mathrm{cf}(\alpha_\nu(\lambda))=\mathrm{cf}(\lambda)$ *for any $\lambda \in \Phi_\nu^l$.*

Let λ be a limit number less than $\beta^\#$. According to Theorem 2 iv), there exists the least number ν, which we denote by $\mathrm{gn}(\lambda)$ (*genus* of λ), such that $\lambda<\alpha_\nu(\omega_\beta^\lambda)$, and according to Theorem 2 iii), there exists a $\mu\in\Phi_\nu^l$, which we denote by $\mathrm{de}(\lambda)$ (*derivation* of λ) such that $\lambda=\alpha_\nu(\mu)$. According to Theorem 2 ii), $\mathrm{de}(\lambda)$ is uniquely determined by λ. For an isolated number λ, put $\mathrm{gn}(\lambda)=\mathrm{gn}(\mathrm{ls}(\lambda))$ and $\mathrm{de}(\lambda)=\mathrm{de}(\mathrm{ls}(\lambda))$. Let $\gamma(\lambda)$ denote the least number ξ such that $\lambda<\xi\omega$. A number λ such that $\gamma(\lambda)=\lambda$ is called a *γ-number* (see [1]).

DEFINITION 4. *Γ^0 denotes the set of all $\lambda<\beta^\#$ with $\mathrm{cf}(\mathrm{ls}(\lambda))<\beta$. Γ^1 denotes the set of all $\lambda<\beta^\#$ such that $\mathrm{cf}(\mathrm{ls}(\lambda))=\beta$ and $\mathrm{de}(\lambda)$ is not a γ-number. Γ^2 denotes the set of all $\lambda<\beta^\#$ such that $\mathrm{cf}(\mathrm{ls}(\lambda))=\beta$ and $\mathrm{de}(\lambda)$ is a γ-number.*

By Definition 4 and Theorem 2 v), we can easily see

LEMMA 1. *$\lambda\in\Gamma^1$ uniquely determines ν,ξ,ζ and n such that $\nu=\mathrm{gn}(\lambda)$, $\xi=\gamma(\mathrm{de}(\lambda))$, $n=\mathrm{fr}(\lambda)$ and (1) $\lambda=\alpha_\nu(\xi+\zeta)+n$. $\lambda\in\Gamma^2$ uniquely determines ν,μ and n such that $\nu=\mathrm{gn}(\lambda)$, $n=\mathrm{fr}(\lambda)$ and (2) $\lambda=\alpha_\nu(\omega_\beta^\mu)+n$, where μ is either an isolated number or a limit number with $\mathrm{cf}(\mu)=\beta$.*

DEFINITION 5. *The right sides of (1) and (2) in Lemma 1 are called the canonical decompositions of λ in Γ^1 and Γ^2 respectively.*

3. Outline of the proof of Theorem A. W_λ denotes the set of all ordinal numbers $\mu<\lambda$ with the natural order between them. Let $\{X_\lambda\mid\lambda\in\Lambda\}$ be a family of ramified sets.

DEFINITION 6. i) *$\bigvee_{\lambda\in\Lambda}X_\lambda$ denotes the cardinal sum of sets X_λ with $\lambda\in\Lambda$, and $W_\lambda+X$ denotes the ordinal sum of W_λ and X* (see [2]). ii) *$\mathrm{Seg}_\nu(X)$ denotes the set of all $x\in X$ such that the type, which is an ordinal number, of $\mathrm{Lb}(x)$ is less than ω^ν. $X\odot_\nu Y$ denotes the set of all $x\in X$ and all pairs (x,y) with $x\in\mathrm{Seg}_\nu(X)$ and $y\in Y$ where the order within X preserves original relation, $x<(x',y)$ if and only if $x\leq x'$ within X and $(x,y)<(x',y')$ if and only if $x=x'$ and $y<y'$ within Y.*

According to Axiom of Choice, we assume that for any limit number λ a sequence Λ_λ of ordinal numbers less than and cofinal to λ is determined so that $\overline{\overline{\Lambda}}_\lambda=\aleph_{\mathrm{cf}(\lambda)}$. For an isolated number λ, put $\Lambda_\lambda=\Lambda_{\mathrm{ls}(\lambda)}$. Besides put $\sigma_\xi=\omega$ if $\xi\in\Gamma^1\cup\Gamma^2$ and $\sigma_\xi=0$ if $\xi\in\Gamma^0$. In order to prove the Main Theorem A, we shall define a sequence $\mathfrak{N}=\{N_\lambda\mid\lambda<\beta^\#\}$ of $N_\lambda\in\mathfrak{S}_\beta$ according to

PRINCIPLE 1. *Put $N_n=W_n$ for $n<\omega$, and assume $\omega\leq\lambda<\beta^\#$ and that for any $\mu<\lambda$, N_μ is already defined.*

Case $\lambda\in\Gamma^0$. Put $N_\lambda=W_n+\bigvee_{\mu\in\Lambda_\lambda}N_\mu$ where $n=\mathrm{fr}(\lambda)$.

Case $\lambda\in\Gamma^1$. Let $\alpha_\nu(\xi+\zeta)+n$ be the canonical decomposition of λ, and put $N_\lambda=N_{\alpha_\nu(\zeta)+n}\odot_\nu N_{\alpha_\nu(\xi)+\sigma_\xi}$.

Case $\lambda\in\Gamma^2$. Let $\alpha_\nu(\omega_\beta^\mu)+n$ be the canonical decomposition of λ.

If μ is a limit number with $\mathrm{cf}(\mu)=\beta$, *then put* $N_\lambda=W_{\omega^\nu}+N_{\mu+n}$, *and if* $\mu=\zeta+1$, *then put* $N_\lambda=W_{\omega^\nu}+N_{\omega\zeta+\sigma_\zeta+n}$.

Then we can prove

THEOREM 3. i) *For any* $X\varepsilon\mathfrak{R}_\beta$, *there exists a* $\lambda<\beta^{\#}$ *such that* $X\propto N_\lambda$. ii) *Each* N_λ *in* $\mathfrak{N}$ *satisfies the following conditions:* D.1) $\mu<\lambda$ *implies* $N_\mu\precnsim N_\lambda$, D.2) *If* $X\varepsilon\mathfrak{S}_\beta$ *and* $N_\mu\precnsim X$ *for any* $\mu<\lambda$ *then* $N_\lambda\propto X$, *and* D.3) *If* $X\varepsilon\mathfrak{R}_\beta$, *then* N_λ *is comparable with* X.

From this we have the

COROLLARY. i) *For any* $X\varepsilon\mathfrak{S}_\beta$ *there exists a* $\lambda<\beta^{\#}$ *such that* $X\sim N_\lambda$. ii) *Under identification of equivalent sets,* $\mathfrak{S}_\beta$ *is well-ordered by* $\propto$ *in the type* $\beta^{\#}$ (*Main Theorem* A i)).

Since, for any ramified sets X and Y, there exists a sufficiently large regular number $\beta>0$ such that both $\overline{\overline{X}}$ and $\overline{\overline{Y}}$ are less than $\aleph_\beta$, Main Theorem A ii) follows from the above Corollary i) and D.3) on N_λ. Thus the proof of Main Theorem A is completed.

4. Sets S_ζ^ν. DEFINITION 7. *Let* $P_n^\nu(X)$ *denote the set of all sequences* $p=\{p(1),p(2),\cdots,p(n)\}$ *of length* n, *which we denote by* $\mathrm{len}(p)$, *such that* $p(k)\varepsilon\mathrm{Seg}_\nu(X)$ *for* $k<n$ *and* $p(n)\varepsilon X$. $X^{\circledast_\nu}$ *denotes the set* $\bigcup_{n<\omega}P_n^\nu(X)$, *in which for* p *and* p', *we have* $p<p'$ *if and only if, putting* $n=\mathrm{len}(p)$ *and* $n'=\mathrm{len}(p')$, *one of the following conditions is satisfied:* (a) $n<n'$, $p(k)=p'(k)$ *for any* $k<n$ *and* $p(n)\leq p'(n)$, (b) $n=n'$, $p(k)=p'(k)$ *for any* $k<n$ *and* $p(n)<p'(n)$.

DEFINITION 8. *Put* $\zeta'=\omega_\beta^\zeta$ *and* $\sigma'_\zeta=\sigma_{\zeta'}$, i.e. $\sigma'_\zeta=0$ *if and only if* ζ *is a limit number with* $\mathrm{cf}(\zeta)<\beta$ *and* $\sigma'_\zeta=\omega$ *otherwise. Let* ν *be a number with* $0<\nu<\omega_\beta$. *Put* $S_0^\nu=N^{\circledast_\nu}_{\alpha_\nu(\nu)}$ *and* $S_\zeta^\nu=N^{\circledast_\nu}_{\alpha_\nu(\omega_\beta^\zeta)+\sigma'_\zeta}$ *for any* ζ *with* $0<\zeta<\beta^{\#}$.

Then for any ζ with $0\leq\zeta<\beta^{\#}$, we have

THEOREM 4. i) $N_\mu\precnsim S_\zeta^\nu$ *for any* $\mu<\alpha_\nu(\omega_\beta^{\zeta+1})$. ii) *If* $X\varepsilon\mathfrak{R}_\beta$ *and* $N_\mu\propto X$ *for any* $\mu<\alpha_\nu(\omega_\beta^{\zeta+1})$, *then* $S_\zeta^\nu\propto X$. iii) *If* $X\varepsilon\mathfrak{R}_\beta$, *then* S_ζ^ν *is comparable with* X. iv) $S_\zeta^\nu\precnsim N_{\alpha_\nu(\omega_\beta^{\zeta+1})}$ *except where* $\zeta=0$ *and* ν *is either an isolated number or a limit number with* $\mathrm{cf}(\nu)=0$. $S_\zeta^\nu\sim N_{\alpha_\nu(\omega_\beta^{\zeta+1})}$ *in these excepted cases.*

Thus S_ζ^ν is the least upper bound of sets N_μ with $\mu<\alpha_\nu(\omega_\beta^{\zeta+1})$ within $\mathfrak{R}_\beta$. We shall find examples to confirm Main Theorem B from the family $\{X\,|\,X\varepsilon\mathfrak{R}_\beta,\ S_1^1\precnsim X\precnsim N_{\omega_\beta^2}\}$.

5. Outline of the proof of the Main Theorem B

DEFINITION 9. *Put* $L=\bigvee_{3\leq n<\omega}N_{\omega_\beta+n}(=\bigvee_{3\leq n<\omega}W_{\omega+n}\sim N_{\omega_\beta+\omega})$ *and* $S=L^{\circledast_1}$. *Let* $\mathfrak{A}$ *be the family of all maximal totally ordered subsets* A *of* S *such that* $A\frown P_n^1(L)$ *is not void for any* $n<\omega$ (*then* $\bar{A}=\omega$), *and* B *be the set of elements* t_A *assigned to each* $A\varepsilon\mathfrak{A}$. *Put* $T=S\smile B$ *where each* $t\varepsilon B$ *is maximal in* T *and* $x<t_A$ *for* $x\varepsilon S$ *if and only if* $x\varepsilon A$. *For a*

subset C of B, $S^{\smile}C$ is regarded as a ramified subset of T.

LEMMA 2. $L \precnapprox S \precnapprox T$.

A subset C of B will be called *barren* if $S^{\smile}C \propto S$. Applying Proposition P_8 of [4] we can see

THEOREM 5. *Under the assumption of Continuum Hypothesis, for any subset C of B, if it is not barren, there exists a subset D of C such that $S \precnapprox S^{\smile}D \precnapprox S^{\smile}C$.*

By Lemma 2, B itself is not barren, and starting from $D_0=B$, we can inductively define D_n such that each D_n is not barren and $S^{\smile}D_{n+1} \precnapprox S^{\smile}D_n$ for any $n<\omega$. Hence putting $X_n=S^{\smile}D_n$, the sequence X_n, $n<\omega$, satisfies the condition of Main Theorem B ii), and we obtain the proof of it.

Let D be a subset of B such that $S \precnapprox S^{\smile}D \precnapprox T$, and E be the set of elements s_t assigned to each $t \varepsilon D$. Put $U=S^{\smile}D^{\smile}E$ where each $s \varepsilon E$ is maximal in U and $x<s_t$ for $x \varepsilon S^{\smile}D$ if and only if $x \leq t$ within $S^{\smile}D$. Then we can see that neither $T \propto U$ nor $U \propto T$ holds, and we obtain the proof of Main Theorem B i).

6. Outline of the proof of the Main Theorem C. Hereafter we assume $\beta=1$.

DEFINITION 10. *Let Δ_0 and Δ_1 be the sets of numbers less than $1^{\#}$ such that: If $\lambda \varepsilon \Gamma^0$, then $\lambda \varepsilon \Delta_0$. Assume that $\lambda<1^{\#}$ and it is decided whether $\mu \varepsilon \Delta_0$ or $\mu \varepsilon \Delta_1$ for any $\mu<\lambda$. For $\lambda \varepsilon \Gamma^1$, letting $\alpha_\nu(\xi+\zeta)+n$ be the canonical decomposition of λ, $\lambda \varepsilon \Delta_0$ or $\lambda \varepsilon \Delta_1$ according to $\alpha_\nu(\zeta) \varepsilon \Delta_0$ or $\alpha_\nu(\zeta) \varepsilon \Delta_1$ respectively. For $\lambda \varepsilon \Gamma^2$, letting $\alpha_\nu(\omega_1^\mu)+n$ be the canonical decomposition of λ, $\lambda \varepsilon \Delta_0$ if $\mu=1$, $\lambda \varepsilon \Delta_1$ if $\mu=\zeta+1$ where $\zeta>0$ and, if μ is a limit number, then $\lambda \varepsilon \Delta_0$ or $\lambda \varepsilon \Delta_1$ according to $\mu \varepsilon \Delta_0$ or $\mu \varepsilon \Delta_1$ respectively.*

Then any number less than $1^{\#}$ is shared into Δ_0 or Δ_1. In order to prove Main Theorem C, we shall construct a sequence $\mathfrak{M}=\{M_\lambda \mid \lambda<1^{\#}\}$ of $M_\lambda \varepsilon \mathfrak{R}_1$ according to

PRINCIPLE 2. *If $\lambda \varepsilon \Delta_0$, then $M_\lambda=N_\lambda$. If $\lambda \varepsilon \Delta_1$ and $\lambda=\mu+1$, then $M_\lambda=N_\mu$. If $\lambda \varepsilon \Delta_1 \frown \Gamma^2$, $\mathrm{fr}(\lambda)=0$ and, letting $\alpha_\nu(\omega_1^\mu)$ be the canonical decomposition of λ, $\mu=\zeta+1$ where $\zeta>0$, then $M_\lambda=S_\zeta^\nu$.*

Now assume $\lambda \varepsilon \Delta_1$, $\mathrm{fr}(\lambda)=0$ and that M_δ is defined for any $\delta<\lambda$. If $\lambda \varepsilon \Gamma^1$, then letting $\alpha_\nu(\xi+\zeta)$ be the canonical decomposition of λ, $M_\lambda=M_{\alpha_\nu(\zeta)} \odot_\nu N_{\alpha_\nu(\xi)+\sigma_\zeta}$. If $\lambda \varepsilon \Gamma^2$ and, letting $\alpha_\nu(\omega_1^\mu)$ be the canonical decomposition of λ, μ is a limit number, then $M_\lambda=W_{\omega^\nu}+M_\mu$.

Thus we can inductively define M_λ for any $\lambda<1^{\#}$, and we have

THEOREM 6. i) *For any $X \varepsilon \mathfrak{R}_1$, there exists a $\lambda<1^{\#}$ such that $X \propto M_\lambda$.* ii) *Each M_λ satisfies* D.1) *$\mu<\lambda$ implies $M_\mu \precnapprox M_\lambda$ and* D.2′) *If $X \varepsilon \mathfrak{R}_1$ and $M_\mu \precnapprox X$ for any $\mu<\lambda$, then $M_\lambda \propto X$.*

From this, Main Theorem C follows similarly as Main Theorem A i) follows from Theorem 3.

References

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