118. Characterizations of Spaces with Dual Spaces. II

By Takesi ISIWATA
Tokyo Gakugei University
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We have proved in [1] that, for a completely regular space X, the following conditions are equivalent: i) X is a stonean space with a dual space, ii) any proper open subspace U of X has a dual space and $X-U$ is inessential to X*) and iii) any proper dense subspace of X has a dual space. For a completely regular space with a dual space, its characterizations are given in [2]. For instance, we have proved that X has a dual space if and only if any proper open subspace U of X whose complement is compact has a dual space. This is a generalization of iii) mentioned above because the subspace U whose complement is compact is dense in X if X has a dual space.

In this paper, we shall first generalize ii) mentioned above, that is, we shall show that X has a dual space if and only if any proper open subspace U of X has a dual space. But this does not mean that a dual space of U is $\overline{U}(\text{in } \beta X)-U$. If U has always $\overline{U}(\text{in } \beta X)-U$ as a dual space, then X becomes a stonean space with a dual space. Next, since examples of spaces with dual spaces given in [2] are all pseudo-compact, we shall give here non-pseudo-compact spaces with non-pseudo-compact dual spaces.

1. We shall first state a useful lemma.

Lemma 1. *Let F be a closed subset of X, if f is a bounded continuous function on $X-F$, then f has a continuous extension over $\beta X-\overline{F}(\text{in } \beta X)$.*

A proof of this lemma follows from the proof of ii) of Lemma 1 in [2].

Theorem 1. *X has a dual space if and only if any proper open subspace has a dual space.*

Proof. If any open subspace has a dual space, then X has a dual space by i)$\leftrightarrow$ii) of Theorem 2 in [2]. Thus, to prove the theorem, it is sufficient to show the converse. Suppose that X has a dual space. If U is open X, any point of U has no compact neighborhoods and any bounded continuous function on U can be continuously extended over M where $M=X^*-E$, $E=\overline{(X-U)}(\text{in } \beta X)$ by Lemma 1. It is easily seen that $E\frown X=X-U$ and M is an open subspace of X^* whose points have no compact neighborhoods. Now we consider the Stone-

*) See footnote 2) in [2].

Čech compactification Z of $U \cup M$ and we put $V=Z-U$. Then both spaces U and V are dense in Z and every point of U (or of V) has no compact neighborhoods. We shall show that U has a dual space V. To do this, it is sufficient to show that any bounded continuous function on U (or on V) has a continuous extension over V (or U). A bounded continuous function f on U can be continuously extended over M by Lemma 1, and hence over Z because Z is the Stone-Čech compactification of $U \cup M$. Conversely a bounded continuous function g on V is continuous on M. By Lemma 1 and by that X^* has a dual space X, g has a continuous extension over $U \cup M$, and hence over Z. We denote by h this extension (over Z) of g. It is obvious that h is a continuous extension of g over V. Thus U has a dual space V.

2. In this section, we shall show the existence of a space X with a dual space X^* such that both spaces X and X^* are not pseudo-compact.

If X is not discrete, there is a non-isolated point x. For a given neighborhood W of x, by the standard construction method of a continuous function for a completely regular space, we can construct a continuous function f such that $f=1$ on $X-W$, $f(x)=0$ and $f(x_n)\to 0$ for some sequence $\{x_n\}$. Let U be an inverse image by f of an open interval $(0,1)$. Then U is a proper open subspace of X and $1/f$ is an unbounded continuous function on U. This means that U is not pseudo-compact. In this case, if X has a dual space, then by Theorem 1, U has a dual space U^*. But U^* may be pseudo-compact. Thus, in the following, we assume that X is a non-pseudo-compact space with a dual space X^* where X^* may be pseudo-compact, and *we shall construct a subspace Y of βX such that* i) *Y has a dual space,* ii) *$\beta X(=Z)=Y \cup Y^*$, $Y \cap Y^*=\theta$ and* iii) *both spaces Y and Y^* are not pseudo-compact.*

Since X is not pseudo-compact, there is a family $\{U_n^1;\ n=1,2,\cdots\}$ of open sets of X such that i) $\{U_n^1\}$ is locally finite, ii) $\overline{U_n^1}$ (in X) $\cap \overline{U_m^1}$ (in X)$=\theta$ and iii) every U_n^1 is regularly open, that is, U_n^1 is equal to an open kernel of $\overline{U_n^1}$ (in X) (symbolically, $U_n^1=\mathrm{Int}_X(\mathrm{Cl}_X U_n^1)$). Now let us put $U_n=\mathrm{Int}_Z(\mathrm{Cl}_Z U_n^1)$. Then U_n is regularly open in Z, and hence $U_n^2=X^* \cap U_n$ is also regularly open in X^* (for instance, see [3]).

a) *$U_n^2 \cap U_m^2=\theta$ for $m \neq n$ (and hence $U_n \cap U_m=\theta$).*

Suppose that $U_n^2 \cap U_m^2 \ni a$ $(n \neq m)$. Then there is a regularly open subset W in Z containing the point a such that $U_n \cap U_m \supset W$. This implies that $U_n^1 \cap U_m^1 \supset X \cap W \neq \theta$ because X is dense in Z. This is a contradiction.

By the assumption on $\{U_n^1\}$, $\mathrm{Cl}_X\left(\bigcup_{n=1}^{\infty} U_n^1\right)-\left(\bigcup_{n=1}^{\infty} U_n^1\right)=\theta$. On the other hand, we have $A=\mathrm{Cl}_Z\left(\bigcup_{n=1}^{\infty} U_n\right)-\left(\bigcup_{n=1}^{\infty} U_n\right)\neq\theta$ from the compactness of Z. Now suppose that $A\frown X\ni b$, that is, any neighborhood W (in Z) of b intersects infinitely many U_n. This means that $W\frown X$ is a neighborhood of b which intersects infinitely many U_n^1. This is a contradiction, thus we have

b) *A is a compact subset contained in X^*.*

Moreover, by the method of construction of $\{U_n^2\}$, we have

c) $A=\mathrm{Cl}_{X^*}\left(\bigcup_{n=1}^{\infty} U_n^2\right)-\left(\bigcup_{n=1}^{\infty} U_n^2\right)$.

Thus A is considered as a subset of X^*. Since every point of X^* has no compact neighborhood (in X^*), every point of A has also no compact neighborhoods (in X^*), thus we have

d) *A has no inner points (as a subspace of X^*).*

Let us put $Y=X\smile A$ and $Y^*=X^*-A$. We shall show that Y and Y^* are the desired spaces.

For a bounded continuous function f on Y, $f|X$ is continuous on X, and hence $f|X$ has a continuous extension over βX. It is obvious that this extension coincides with f on Y. Conversely, let g be a bounded continuous function on Y^*. By Lemma 1, g has a continuous extension g_1 over $\beta X-A$. Since $\beta X-A\supset X$ and X has a dual space X^*, g_1 has a continuous extension over βX. This means that g is continuously extended over Y. Since both spaces Y and Y^* have no points with compact neighborhoods, Y is a space with a dual space Y^*, and both spaces Y and Y^* are not pseudo-compact.

References

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