115. On Generalized Peano's Theorem concerning the Dirichlet Problem for Semi-linear Elliptic Differential Equations

By Isamu HIRAI and Kiyoshi AKÔ

(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1960)

The purpose of this note is to prove a theorem which concerns the Dirichlet problem for semi-linear elliptic differential equations and is similar to Peano's theorem concerning the initial value problem of ordinary differential equations of the first order.[1] The precise statement of the theorem will be given in §2.

The authors of this note wish to express their deepest gratitude to Prof. Masuo Hukuhara who has constantly inspired and stimulated them.

1. **Preliminaries.** In this note we shall consider the semi-linear elliptic differential equation

$$(1) \qquad \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u) \text{ [2]}$$

in a bounded domain G under the following assumptions.

Assumptions. 1°. G is a bounded Poincaré domain in the Euclidean m-space; i.e. for each boundary point x of G there exist one half C_x of a circular cone with vertex x and a closed sphere K_x with center x such that

$$C_x \frown K_x \frown \overline{G} = \{x\}.\text{[3]}$$

2°. The symmetric matrix $\|a_{ij}(x)\|$ is continuous and positive-definite in the closure $\overline{G}$ of G.

3°. The function $f(x, u, p)$ $(p=(p_1, \cdots, p_m))$ is defined in $\mathfrak{D}: x \in \overline{G}$, $|u| < \infty$, $|p| < \infty$ and Hölder-continuous (with some exponent α, $0 < \alpha < 1$) in every compact subset of $\mathfrak{D}$. Further $f(x, u, p)$ is non-decreasing with respect to u; i.e.

$$f(x, u, p) \leqq f(x, \overline{u}, p) \quad \text{provided } x \in \overline{G},\ u < \overline{u},\ |p| < \infty.$$

Moreover, we assume that for every constant $M > 0$ there exist two constants $B(M)$ and $F(M)$ such that

$$|f(x, u, p)| \leqq B(M)|p| + F(M)$$

1) As for generalized Peano's theorem concerning the Dirichlet problem see T. Satō: Sur l'equation aux derivées partielles $\Delta z = f(x, y, z, p, q)$ I, Compositio Math., **12**, 157–177 (1954); II, ibid., **14**, 152–172 (1959). See, in particular, Théorème 3 of the second note.

2) Here $x = (x_1, \cdots, x_m)$ and grad $u = (\partial u/\partial x_1, \cdots, \partial u/\partial x_m)$.

3) We denote by $\overline{G}$ the closure $G + \Gamma$ of the domain G, where Γ is the boundary of G.

provided that $x\in\overline{G}$, $|u|\leqq M$, $|p|<\infty$.

4°. The $a_{ij}(x)$ satisfy the Lipschitz condition in $\overline{G}$;

$$\sum_{i,j=1}^{m}|a_{ij}(x)-a_{ij}(y)|\leqq L|x-y| \quad \text{(for some constant } L>0).$$

For the sake of convenience we give here the following

Definition. We denote by C^* the class of functions $u(x)$ satisfying the following two conditions:

a) $u(x)$ is in $C^2(G)\frown C(\overline{G})$,

b) $||u||_S^{\alpha,2}$ is finite for any closed sphere S contained in the domain G, where

$$||u||_S^{\alpha,2}\equiv\sup_{x\in S}\left[|u(x)|+\sum_{k=1}^{m}\left|\frac{\partial u}{\partial x_k}\right|+\sum_{i,j=1}^{m}\left|\frac{\partial^2 u}{\partial x_i\partial x_j}\right|\right]+\sum_{i,j=1}^{m}H_\alpha^S\left(\frac{\partial^2 u}{\partial x_i\partial x_j}\right),$$

$$H_\alpha^S(g)=\sup_{\substack{x,y\in S\\ x\neq y}}\frac{|g(x)-g(y)|}{|x-y|^\alpha} \quad (0<\alpha<1).$$

M. Nagumo proved the following Theorem I in his famous note.[4)]

THEOREM I (*M. Nagumo*)

Assumptions. 1°. *Let $\varphi(x)$ be any continuous function prescribed on the boundary Γ of G.*

2°. *There exist two functions $\underline{\omega}(x)$ and $\overline{\omega}(x)$ in $C(\overline{G})$, where $\underline{\omega}(x)=\max_{\nu=1,2}\underline{\omega}_\nu(x)$ with $\underline{\omega}_\nu(x)\in C^2(G)\frown C(\overline{G})$ which satisfy the conditions*

$$\begin{cases}\displaystyle\sum_{i,j=1}^{m}a_{ij}(x)\frac{\partial^2\underline{\omega}_\nu}{\partial x_i\partial x_j}\geqq f(x,\underline{\omega}_\nu,\operatorname{grad}\underline{\omega}_\nu) & \text{in } G,\\ \underline{\omega}_\nu(x)<\varphi(x) & \text{on the boundary } \Gamma \ (\nu=1,2),\end{cases}$$

and $\overline{\omega}(x)=\min_{\nu=1,2}\overline{\omega}_\nu(x)$ with $\overline{\omega}_\nu(x)\in C^2(G)\frown C(\overline{G})$ which satisfy the conditions

$$\begin{cases}\displaystyle\sum_{i,j=1}^{m}a_{ij}(x)\frac{\partial^2\overline{\omega}_\nu}{\partial x_i\partial x_j}\leqq f(x,\overline{\omega},\operatorname{grad}\overline{\omega}_\nu) & \text{in } G,\\ \overline{\omega}_\nu(x)>\varphi(x) & \text{on the boundary } \Gamma \ (\nu=1,2).\end{cases}$$

Further we assume that $\underline{\omega}(x)<\overline{\omega}(x)$ in the closure $\overline{G}$ of G.

Conclusion. *There exists at least one solution $u(x)$ in C^* of the Dirichlet problem $[D_\varphi]$:*

$$\begin{cases}\displaystyle\sum_{i,j=1}^{m}a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}=f(x,u,\operatorname{grad}u) & \text{in the domain } G,\\ u(x)=\varphi(x) & \text{on the boundary } \Gamma \text{ of } G,\end{cases}$$

satisfying the inequalities

$$\underline{\omega}(x)\leqq u(x)\leqq\overline{\omega}(x) \qquad \text{in the closure } \overline{G} \text{ of } G.$$

We remark here that M. Nagumo obtained a more general existence theorem. But for the sake of convenience we quote his

4) M. Nagumo: On principally linear elliptic differential equations of the second order, Osaka Math. J., **6**, 207-229 (1954) (see, in particular, Theorem 6).

result in this weaker form.

COROLLARY. *We can always construct the functions* $\underline{\omega}(x)$ *and* $\overline{\omega}(x)$ *in the form*

$$\underline{\omega}(x)=-\theta(\theta-e^{-\gamma x_1}) \quad \textit{and} \quad \overline{\omega}(x)=\theta(\theta-e^{-\gamma x_1})$$

for sufficiently large numbers θ, γ. *Hence the Dirichlet problem* $[D_\varphi]$ *is always solvable.*

REMARK 1. If we choose two numbers θ and γ sufficiently large, then $\underline{\omega}(x)$ and $\overline{\omega}(x)$ satisfy the differential inequalities

$$\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 \underline{\omega}}{\partial x_i \partial x_j} \geqq f(x, \underline{\omega}, \operatorname{grad} \underline{\omega})+1$$

$$\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 \overline{\omega}}{\partial x_i \partial x_j} \leqq f(x, \overline{\omega}, \operatorname{grad} \overline{\omega})-1$$

respectively in G, together with the boundary condition

$$\underline{\omega}(x)<\varphi(x)<\overline{\omega}(x) \qquad \text{on the boundary } \Gamma.$$

THEOREM II. *Let* $\{u(x, \delta)\}$ $(-1<\delta<1)$ *be any set of solutions in* C^* *of the Dirichlet problem containing a constant parameter*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{m} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}=f(x, u, \operatorname{grad} u)-\delta & \textit{in } G, \\ u(x)=\varphi(x) & \textit{on the boundary } \Gamma \textit{ of } G. \end{cases}$$

Then there exist two sequences $\delta_n \downarrow 0$ *and* $\delta_{n'} \uparrow 0$ *such that the sequence* $\{u(x, \delta_n)\}$ *converges from above to a solution* $u_{\max}(x)$ *in* C^* *of the Dirichlet problem* $[D_\varphi]$, *and the sequence* $\{u(x, \delta_{n'})\}$ *converges from below to a solution* $u_{\min}(x)$ *in* C^* *of the Dirichlet problem* $[D_\varphi]$. *Further if* $u(x)$ *is any solution in* C^* *of the Dirichlet problem* $[D_\varphi]$, *then*

$$u_{\min}(x)\leqq u(x)\leqq u_{\max}(x) \qquad \textit{in the closure } \overline{G}.$$

REMARK 2. The existence of the family of functions $\{u(x, \delta)\}$ $(-1<\delta<1)$ is a direct consequence of the corollary of Theorem I.

REMARK 3. Let $\underline{\omega}(x)$ and $\overline{\omega}(x)$ be the functions stated in Remark 1. Then it is easily seen that $\underline{\omega}(x)\leqq u(x, \delta)\leqq u(x, \delta')\leqq\overline{\omega}(x)$ in $\overline{G}$ provied that $-1<\delta<\delta'<1$.[5] According to a theorem due to M. Nagumo[6] we see that for every closed sphere S contained in G there exists a constant $K(S)$ such that

$$\| u(*, \delta) \|_S^{\alpha, 2} \leqq K(S)$$

for every $u(x, \delta)$ $(-1<\delta<1)$.

Proof. By virtue of Remark 3 there exists a suitable sequence $\delta_n \downarrow 0$ such that for a certain $v(x)\in C^2(G)$

$$u(x, \delta_n)\to v(x), \qquad \partial u(x, \delta_n)/\partial x_k \to \partial v/\partial x_k$$

5) The proof follows from the fact that if $v(x)\in C^2(G)$ attains the greatest value at a point x_0 in G, then

$$\sum_{i,j=1}^{m} a_{ij}(x_0)\frac{\partial^2 v}{\partial x_i \partial x_j}\leqq 0, \qquad \operatorname{grad} v(x_0)=0.$$

6) See Lemma 3 of the note of M. Nagumo, loc. cit. in the footnote 5).

$$\partial^2 u(x, \delta_n)/\partial x_i \partial x_j \to \partial^2 v/\partial x_i \partial x_j$$

uniformly in every closed sphere S contained in G. The limit function $v(x)$ clearly satisfies the differential equation (1) in G. But since $u(x, -\frac{1}{2}) \leqq v(x) \leqq u(x, \delta_n)$ in G, we see that $v(x)$ is continuous in the closure $\overline{G}$ of G. Hence the function $v(x) = u_{\max}(x)$ is a solution in C^* of the Dirichlet problem $[D_\varphi]$.

In the same way we can see the existence of the solution $u_{\min}(x)$ in C^* of the Dirichlet problem $[D_\varphi]$. The last statement of the theorem follows from the inequalities

$$u(x, \delta_{n'}) \leqq u(x) \leqq u(x, \delta_n)$$

(see Remark 3).

REMARK 4. Thus we have shown the existence of the maximal solution $u_{\max}(x)$ and of the minimal solution $u_{\min}(x)$ of the Dirichlet problem $[D_\varphi]$. We note here that $u_{\max}(x) \neq u_{\min}(x)$ in general. T. Satō gave a concrete example of the "maximal" and the "minimal" solutions in one of his notes.[7] In a forthcoming paper of K. Akô a concrete example for generalized Peano's theorem will be given.[8]

2. Generalized Peano's theorem. In this section we shall state the generalized Peano's theorem in a precise form and prove it.

THEOREM III (*Generalized Peano's theorem*). *Let G be any bounded Poincaré domain in the Euclidean m-space. And let $\varphi(x)$ be a fixed continuous function prescribed on the boundary Γ of G. We denote by $\mathfrak{U}$ the family of all solutions $u(x)$ in $C^2(G) \frown C(\overline{G})$ of the Dirichlet problem $[D_\varphi]$*

$$(1) \qquad \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} = f(x, u, \operatorname{grad} u) \qquad \text{in } G,$$

$$(2) \qquad u(x) = \varphi(x) \qquad \text{on the boundary } \Gamma \text{ of } G,$$

such that $\|u\|_S^{\alpha,2}$ are finite for any closed sphere S contained in the domain G.[9]

Then there exist two solutions $u_{\min}(x)$ and $u_{\max}(x)$ in $\mathfrak{U}$ such that for every $u(x) \in \mathfrak{U}$ the inequalities $u_{\min}(x) \leqq u(x) \leqq u_{\max}(x)$ are valid in the closure $\overline{G}$ of G. And further for $x \in G$ the set

$$l_x \equiv \{u(x);\ u \in \mathfrak{U}\}$$

is either a closed segment or a single point, which means that all other solutions of the Dirichlet problem $[D_\varphi]$ fill the gap between the functions $u_{\min}(x)$ and $u_{\max}(x)$.

Proof. 1°. The existence of the solutions $u_{\min}(x)$ and $u_{\max}(x)$

7) See the first note of T. Satō, loc. cit. in 1).

8) K. Akô: On the Dirichlet problem for quasi-linear elliptic differential equations of the second order, to appear in J. Math. Soc. Japan.

9) i.e. $u(x) \in C^*$, see Definition in §1. We remark here that every $C^2(G) \frown C(\overline{G})$-solution of the Dirichlet problem $[D_\varphi]$ is always in C^*.

has already been verified in Theorem II of the preceding section.

2°. We shall show that if $u_i(x) \in \mathfrak{U}$ $(i=1,2)$ are such that $u_2(x) \leqq u_1(x)$ and if $c = \max_{x \in \bar{G}} (u_1(x) - u_2(x)) \geqq 0$, then there exists an element $u(x) \in \mathfrak{U}$ such that

$$0 \leqq u_1 - u \leqq c/2, \quad \text{and} \quad 0 \leqq u - u_2 \leqq c/2.$$

3°. We set

$$\underline{\omega}_n(x) \equiv \max \{u_2(x) - 1/n,\ u_1(x) - c/2 - 1/n\},$$
$$\bar{\omega}_n(x) \equiv \min \{u_1(x) + 1/n,\ u_2(x) + c/2 + 1/n\}.$$

Then $\underline{\omega}_n(x) < \bar{\omega}_n(x)$ in $\bar{G}$ and further $\underline{\omega}_n(x) < \varphi(x) < \bar{\omega}_n(x)$ on Γ. By virtue of Theorem I there exists a solution $u^{(n)}(x) \in \mathfrak{U}$ satisfying the inequalities

$$\underline{\omega}_1(x) \leqq \underline{\omega}_n(x) \leqq u^{(n)}(x) \leqq \bar{\omega}_n(x) \leqq \bar{\omega}_1(x)$$

in $\bar{G}$. According to Remark 3 there exists a subsequence $\{u^{(n')}(x)\}$ converging to a function $u(x) \in C^2(G)$ uniformly in every closed sphere S contained in G together with first two derivatives. Hence $u(x)$ satisfies the differential equation (1) and since $u_{\min}(x) \leqq u^{(n)}(x) \leqq u_{\max}(x)$, the function $u(x)$ is continuous in the closure $\bar{G}$ of G. Hence $u(x) \in \mathfrak{U}$ and

$$\max (u_2(x), u_1(x) - c/2) \leqq u(x) \leqq \min (u_1(x), u_2(x) + c/2).$$

Thus we have proved the statement of 2°.

4°. According to 3° the closure of the set

$$l'_{x^*} \equiv \{u(x^*);\ u(x) \in \mathfrak{U},\ u_2(x) \leqq u(x) \leqq u_1(x) \text{ in } \bar{G}\}$$

is either a closed segment or a single point. Hence we need only to show that l'_{x^*} is closed in the one-dimensional u-space. Let $u^* \in \bar{l}'_{x^*}$. Then there exists a sequence $\{u_n(x)\}$ in $\mathfrak{U}$ such that $u_n(x^*) \to u^*$. Again by virtue of Remark 3 there exists a subsequence $\{u_{n'}(x)\}$ converging uniformly in every compact subset of G to a function $v(x) \in C^2(G)$ together with its first two derivatives. Hence again as in 3° we see that $v(x) \in \mathfrak{U}$. Evidently the function $v(x)$ satisfies the condition $v(x^*) = u^*$ and therefore $u^* \in l'_{x^*}$, so that l'_{x^*} is closed. Thus we obtain the theorem, setting $u_1(x) \equiv u_{\max}(x)$ and $u_2(x) \equiv u_{\min}(x)$.

We can make precise the theorem as follows.

COROLLARY. *Let $u_i(x)$ $(i=1,2)$ be two solutions of the Dirichlet problem $[D_\varphi]$ (in $\mathfrak{U}$). Further let $u_2(x) \leqq u_1(x)$ in $\bar{G}$. Then for each point x^* of G and for each constant u^* such that $u_2(x^*) \leqq u^* \leqq u_1(x^*)$ there exists a solution $u(x)$ of the Dirichlet problem $[D_\varphi]$ (in $\mathfrak{U}$) satisfying the following two conditions*:

a) $u_2(x) \leqq u(x) \leqq u_1(x)$ *in* $\bar{G}$,

b) $u(x^*) = u^*$ *at the point* x^*.

This corollary follows from 3° and 4° of the proof of the theorem.

COROLLARY. *Let us assume that the solutions of the Dirichlet problem $[D_\varphi]$ are not unique. Then there exists a constant $\varepsilon > 0$ such that the solutions of the Dirichlet problem $[D_\phi]$ are not unique* *provided that* $|\psi(x)-\varphi(x)|<\varepsilon$ *on* Γ.

Proof. Let $\tilde{u}_1(x)$ and $\tilde{u}_2(x)$ are the maximal and the minimal solutions of the Dirichlet problem $[D_\varphi]$ respectively. Further, let ε be a quarter of the maximum of the function $\tilde{u}_1(x)-\tilde{u}_2(x)$ on the closure $\bar{G}$. Clearly ε is a positive constant. Assume that $|\psi(x)-\varphi(x)|<\varepsilon$ on the boundary Γ of G. Then there exist two solutions $u_1(x)$ and $u_2(x)$ of the Dirichlet problem $[D_\psi]$ such that

$$\tilde{u}_1(x)-\varepsilon\leqq u_1(x)\leqq\tilde{u}_1(x)+\varepsilon$$

and

$$\tilde{u}_2(x)-\varepsilon\leqq u_2(x)\leqq\tilde{u}_2(x)+\varepsilon$$

in $\bar{G}$ by virtue of Theorem I. Since at a point x_0 where the function $\tilde{u}_1(x)-\tilde{u}_2(x)$ attains its greatest value we get

$$u_1(x_0)\geqq\tilde{u}_1(x_0)-\varepsilon>\tilde{u}_2(x_0)+\varepsilon\geqq u_2(x_0),$$

we see that two solutions $u_1(x)$ and $u_2(x)$ of the Dirichlet problem $[D_\psi]$ are different. Thus we have established the corollary.