139. *A Problem of Number Theory*

By Kiyoshi ISÉKI

(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1960)

In this paper we shall consider a problem of number theory. In his recent book, Sto Zadan (in Polish), Prof. H. Steinhaus has solved an interesting problem on number theory: For any natural number

$$\alpha=10^{n-1}a_n+10^{n-2}a_{n-1}+\cdots+10^2a_3+10a_2+a_1$$

expressed in the decimal system, we calculate the sum of the squares of its digit of α,

$$\alpha_1=a_n^2+a_{n-1}^2+\cdots+a_3^2+a_2^2+a_1^2.$$

For the number α_1, we calculate the sum of squares of all digits contained in α_1. We repeat the same processes. If we do not reach 1, then we have a cyclic finite sequence:

$$145,\quad 42,\quad 20,\quad 4,\quad 16,\quad 37,\quad 58,\quad 89.$$

This problem is generalised in the following forms: *Let k be a fixed positive integer, for any natural number*

$$\alpha=10^{n-1}a_n+10^{n-2}a_{n-1}+\cdots+10^2a_3+10a_2+a_1$$

we calculate

$$\alpha_1=a_n^k+a_{n-1}^k+\cdots+a_3^k+a_2^k+a_1^k$$

and for the integer α_1, we calculate the sum of k-powers of all digits a_i of α_1. We repeat the processes. We should like to know all *cyclic parts* appeared except the trivial case.

If such a cyclic part has the sequence with l-terms, we call it a cyclic sequence of the *length l for power k*. Then the results by H. Steinhaus are stated as follows: For $k=2$, there appear a cyclic sequence of the length 8 and a trivial sequence of the length 1.

We can prove *theoretically* that there exist finite numbers of cyclic sequences for each power k. We can not find these *individual* cyclic parts by theoretic methods, and this difficulty is therefore purely technical.

Here, we shall decide all cyclic sequences for $k=3$. The detail result will be found in the table, and its calculation was done by a small desk calculator. For $k=3$, it is seen from an easy calculation (see H. Steinhaus, loc. cit.) that we must find all cyclic sequences of numbers less than 2000, and as we can check that new cyclic sequences between 1000 and 2000 do not appear by a trivial verification, the table shows all cyclic parts from 1 to 999.

1	**1**							
2	8	512	134	92	737	713	**371**	
3	27	351	**153**					
4	64	280	520	133	**55**	**250**	**133**	
5	125	134	92	737	713	**371**		
6	216	225	141	66	432	99	1458	
	702	351	**153**					
7	343	118	514	190	730	**370**		
8	512	(2)	**371**					
9	729	1080	513	**153**				
11	2	(2)	**371**					
12	9	(9)	**153**					
13	28	520	133	**55**	**250**	**133**		
14	65	341	92	737	713	**371**		
15	125	225	(6)	**153**				
16	217	352	**160**	**217**	**352**			
17	344	155	251	134	(2)	**371**		
18	513	**153**						
19	730	**370**						
22	16	(16)	**217**	**352**	**160**			
23	35	152	134	(2)	**371**			
24	72	351	**153**					
25	133	**55**	**250**	**133**				
26	224	80	512	**371**				
27	351	**153**						
28	520	133	**55**	**250**	**133**			
29	737	713	**371**					
33	54	189	1242	**153**				
34	91	730	**370**					
35	152	**371**						
36	243	99	1458	702	**153**			
37	**370**							
38	539	881	1025	134	**371**			
39	756	684	792	1080	(18)	**153**		
44	128	512	134	**371**				
45	189	1242	81	(18)	**153**			
46	280	(28)	**55**	**250**	**133**			
47	**407**							
48	576	684	792	1080	(18)	**153**		
49	793	1099	1459	**919**	**1459**			
55	250	133	(4)	**55**	**250**	**133**		
56	341	**371**						
57	468	792	1080	(18)	**153**			
58	637	586	853	496	1009	(19)	**370**	
59	854	701	(17)	**371**				
66	432	99	1458	702	351	**153**		
67	559	979	1801	514	190	(19)	**370**	
68	728	863	755	593	881	1025	134	**371**
69	945	918	1242	81	(18)	**153**		
77	686	944	857	980	1241	74	(47)	**407**
78	855	762	567	684	792	1080	(18)	**153**
79	1072	352	**160**	**217**	**352**			
88	1024	73	**370**					

89	1241	74	407				
99	1458	702	351	153			
111	3	(3)	153				
112	10	(1)	1				
113	29	(29)	371				
114	66	(66)	153				
115	127	217	352	160	217	352	
116	218	512	125	371			
117	345	216	(6)	153			
118	(7)	370					
119	731	371					
122	17	(17)	371				
123	36	(36)	153				
124	73	(37)	370				
125	(5)	371					
126	225	(6)	153				
127	352	160	217	352			
128	512	125	(5)	371			
129	736	153					
133	55	250	133	55			
134	92	(29)	371				
135	153						
136	244	136	244				
137	371						
138	540	(45)	153				
139	757	811	(118)	370			
144	129	(128)	371				
145	370						
146	281	(128)	371				
147	408	(48)	153				
148	577	811	(118)	370			
149	794	1136	245	197	1073	371	
155	251	(5)	371				
156	342	(6)	153				
157	469	370					
158	638	755	593	881	1025	134	371
159	855	762	567	684	792	(57)	153
166	433	(7)	370				
167	560	(56)	371				
168	729	(9)	153				
169	946	370					
177	687	1071	345	216	(6)	153	
178	856	853	664	496	1009	(19)	370
179	1073	371					
188	1025	(125)	371				
189	1242	81	(18)	153			
199	1459	919	1459				
222	24	(24)	153				
223	43	(34)	370				
224	80	(8)	371				
225	141	(114)	153				
226	232	(223)	370				
227	359	881	(188)	371			

228	528	645	405	(45)	**153**			
229	745	532	**160**	**217**	**234**			
233	62	(26)	**371**					
234	99	(99)	**153**					
235	**160**	**217**	**235**					
236	251	(125)	**371**					
237	378	882	1032	(123)	**153**			
238	547	(229)	**160**	**217**	**235**			
239	764	623	(236)	**371**				
244	136	**244**	**136**					
245	(149)	**371**						
246	288	(237)	**153**					
247	415	(145)	**370**					
248	584	701	(17)	**371**				
249	801	(18)	**153**					
255	258	645	405	(45)	**153**			
256	349	820	(28)	**55**	**250**	**133**		
257	476	623	(236)	**371**				
258	645	405	(45)	**153**				
259	862	736	586	853	664	496	1009	
	(19)	**370**						
266	440	(44)	**371**					
267	567	684	792	1080	(18)	**153**		
268	736	586	853	664	496	1009	(19)	**370**
269	953	881	(188)	**371**				
277	694	1009	(19)	**370**				
278	863	(158)	**371**					
279	1080	(18)	**153**					
288	1032	(123)	**153**					
289	1249	802	(28)	**55**	**250**	**133**		
299	1466	497	(149)	**371**				
333	81	(18)	**153**					
334	(7)	**370**						
335	179	1073	**371**					
336	270	(27)	**153**					
337	397	1099	(199)	**919**	**1459**			
338	566	557	593	881	(188)	**371**		
339	783	882	(288)	**153**				
344	(17)	**371**						
345	**153**							
346	307	(37)	**370**					
347	434	(344)	**371**					
348	603	(36)	**153**					
349	820	(28)	**55**	**250**	**133**			
355	277	(277)	**370**					
356	368	(158)	**371**					
357	495	918	1242	81	(18)	**153**		
358	664	496	1009	(19)	**370**			
359	881	(188)	371					
366	459	918	1242	81	(18)	**153**		
367	586	853	664	496	1009	(19)	**370**	
368	(158)	**371**						
369	972	(279)	**153**					

377	713	**371**					
378	882	(288)	**153**				
379	1099	(199)	**919**	**1459**			
388	1051	(115)	**217**	**352**	**160**		
389	1268	737	(377)	**371**			
399	1485	702	(27)	**153**			
444	192	(129)	**153**				
445	253	(235)	**160**	**217**	**235**		
446	344	(344)	**371**				
447	471	(147)	**153**				
448	640	(46)	**55**	**250**	**133**		
449	857	980	(89)	**407**			
455	314	(134)	**371**				
456	405	(45)	**153**				
457	(229)	**160**	**217**	**235**			
458	701	(17)	**371**				
459	918	1242	81	(18)	**153**		
466	496	1009	(19)	**370**			
467	623	(236)	**371**				
468	792	1080	(18)	**153**			
469	1009	(19)	**370**				
477	750	(57)	**153**				
478	919	(199)	**919**	**1459**			
479	1136	(149)	**371**				
488	1088	(188)	**371**				
489	1305	(135)	**153**				
499	1522	142	73	(37)	**370**		
555	375	495	918	1242	81	(18)	**153**
556	466	(466)	**370**				
557	593	881	(188)	**371**			
558	762	567	684	792	1080	(18)	**153**
559	979	1801	514	190	(19)	**370**	
566	557	(557)	**371**				
567	684	792	1080	(18)	**153**		
568	853	664	496	1009	(19)	**370**	
569	1070	344	155	251	**371**		
577	811	(118)	**370**				
578	980	(89)	**407**				
579	1197	1074	408	576	(567)	**153**	
588	1149	795	1197	1074	408	(48)	**153**
589	1366	460	(46)	**55**	**250**	**133**	
599	1583	665	(566)	**371**			
666	648	792	1080	(18)	**153**		
667	775	811	514	190	(19)	**370**	
668	944	857	980	(89)	**407**		
669	1161	219	882	1032	36	(36)	**153**
677	902	(29)	**153**				
678	1071	345	216	(6)	**153**		
679	1288	1033	**55**	**250**	**133**		
687	1071	(117)	**153**				
688	1240	(124)	**370**				
689	1457	533	179	1073	**371**		
699	1674	624	251	134	**371**		

777	1029	738	882	1032	36	(36)	**153**
778	1198	1243	100	**1**			
779	1415	191	731	**371**			
788	1367	587	980	(89)	**407**		
789	1584	702	(27)	**153**			
799	1801	514	190	(19)	**370**		
888	1536	369	972	1080	(18)	**153**	
889	1753	496	1009	(19)	**370**		
899	1970	1073	**371**				
999	2187	864	792	1080	(18)	**153**	

Therefore, from the table, we have the following cyclic sequences for $k=3$, self-closed sequences (with length 1) are:

1, 153, 370, 371, 407,

cyclic sequences with length 2 are:

136, 244; 919, 1459,

and cyclic sequences with length 3 are:

55, 250, 133; 160, 217, 352.

We are now planning the calculations for the finding of cyclic parts for $k=4, 5, \cdots$. As the calculation is very much complicated, we shall use high-speed automatic computers for the purpose. Its detail will appear in later papers with K. Chikawa and T. Kusakabe.