137. A Remark on the Unique Continuation Theorem for Certain Fourth Order Elliptic Equations

By Taira SHIROTA
Osaka University

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1. Unique continuation theorems for solutions of certain fourth order elliptic equations, which are iterations of two second order elliptic equations, are considered by R. N. Pederson [4], S. Mizohata [3] and L. Hörmander [2].

Here we prove the following results with weaker vanishing requirements than these authors.

Theorem 1. *Let $L^{(i)}(x,D)$ $(i=1,2)$ be homogeneous, second order elliptic operators with coefficients of class C^2 in a neighbourhood G of the origin in Euclidean n-space such that $L^{(1)}(0,\xi)=L^{(2)}(0,\xi)$. Let $L(x,\xi)=L^{(1)}(x,\xi)L^{(2)}(x,\xi)$. If a function $u(x)$ of class C^4 in G satisfies the following two conditions:*

(1.1) *for any $\alpha>0$*

$$\lim_{r\to 0}\Big\{\sum_{|\beta|\leqq 4}|D^\beta u|\Big\}r^{-\alpha}=0,$$

(1.2) *for a positive number M*

$$|L(x,D)u(x)|^2\leqq M\Big\{|u(x)|^2r^{-6}+\sum_{|\beta|=1}|D^\beta u(x)|^2r^{-4}+\sum_{|\beta|=2}|D^\beta u(x)|^2r^{-2}+\sum_{|\beta|=3}|D^\beta u(x)|^2\Big\}\qquad (x\in G),$$

then $u(x)$ identically vanishes in a neighbourhood of the origin.

The proof is based on the method used by H. O. Cordes [1] and R. N. Pederson [4], but we use only the transformation $s=r\int_0^r(e^{-m_0\tau}-1)\frac{1}{\tau}d\tau$. The result was suggested by Professor H. Yamabe and Dr. S. Ito.

2. Let $K^{(m)}(R_1)$ be a class of functions $u(x)$ satisfying the following three conditions:

(2.1) $u(x)$ is defined in a cubic neighbourhood G of the origin with radius R and is in class $C^m(G)$, for any $\alpha>0$

$$\lim_{r\to 0}\Big\{\sum_{|\beta|\leqq m}|D^\beta u|\Big\}r^{-\alpha}=0, \tag{2.2}$$

(2.3) $u(x)=0$ for any x such that $|x|\geqq R_1$ $(R_1<R)$.

Lemma 1. *Let L be an elliptic operator of order 2 represented by polar coordinate systems such that*

(2.4) $$L(u)=u_{|r|r}+\frac{n-1}{r}u_{|r}+\frac{1}{r^2}Nu+\frac{1}{r}(b_r u_{|r})_{|r}+\lambda^{-1}\left(\lambda\frac{1}{r}b_\rho u_{|r}\right)_{|\rho}$$

(2.5) $$Nu=\lambda^{-1}(\lambda\,\overline{a_{\sigma\rho}}u_{|\rho})_{|\sigma},$$

where $a_{ij}(0)=\delta_{ij}$,

$$\overline{a_{\sigma\rho}}=a_{ij}\,\theta_\sigma/x_i\,\theta_\rho/x_j/a_{ij}\frac{x_i}{r}\frac{x_j}{r}$$

$$b_\sigma=a_{ij}x_i\theta_\sigma/x_j/a_{ij}\frac{x_i}{r}\frac{x_j}{r}$$

$$\overline{a_{\sigma\rho|r}}\xi_\sigma\xi_\rho\geqq 2m_0\overline{a_{\sigma\rho}}\xi_\sigma\xi_\rho\geqq|\xi|^2$$

$$\lambda(x)=\frac{\partial O_1}{\partial\theta_1\partial\theta_2\cdots\partial\theta_{n-1}}$$

for any real vector $\xi=(\xi_1,\xi_2,\cdots,\xi_{n-1})$. *Then there are constants* k_1 *and* α_0, *depending only on* n *and also on* $\bar{a}_{\sigma\rho}$ (*in particular* m_0) *respectively such that for any* $u\in K^{(2)}(R_1)$ *and for* $\alpha>\alpha_0$

$$\iint|L(u)|^2r^{-2\alpha+3}e^{-m_0r}dO_1dr\geqq k_1m_0\alpha^3\iint|u|^2r^{-2\alpha}e^{-m_0r}dO_1,dr,$$

where dO_1 *is the usual unit surface element.*

Lemma 1 is proved by the same method used in our previous paper [5].

Let $L^{(l)}$ $(l=1,2)$ be a second order elliptic operator such that $L^{(l)}(u)=a_{ij}^{(l)}\dfrac{\partial^2u}{\partial x_i\partial x_j}$ in G, where $((a_{ij}^{(l)}))$ is positive definite matrix whose elements are functions of class $C^2(G)$. Let k_i be a constant depending only on n and $a_{ij}^{(l)}$ and let α_0 and R_1 be constants depending also on m_0. Then using Cordes' transformation, by Lemma 1 we see the following

Lemma 2. *There are positive constants* k_2, R_1 *and* α_0 *such that for any* $u\in K^2(R_1)$ *and for* $\alpha>\alpha_0$

(A) $$\int|L^{(l)}(u)|^2r^{-\alpha+3}e^{\alpha\phi(r)}dx\geqq k_2m_0\left\{\alpha^3\int|u|^2r^{-\alpha}e^{\alpha\phi(r)}dx+\alpha\int|\nabla u|^2r^{-\alpha+2}e^{\alpha\phi(r)}dx\right\}\qquad(l=1,2),$$

where $$\phi(r)=\int_0^r(1-e^{-m_0t})\frac{dt}{t}.$$

3. On the other hand, using Pederson's consideration [4, § 2, Lemma 1], we see the following

Lemma 3. *There are constants* k_3, R_1 *and* α_0 *such that for any* $u(x)\in K^{(2)}(R_1)$ *and for* $\alpha>\alpha_0$

$$\sum_{|\beta|=2}\int|D^\beta u|^2r^{-\alpha}e^{\alpha\phi(r)}dx\leqq\int|\Delta u|^2r^{-\alpha}e^{\alpha\phi(r)}dx+k_3\alpha^2\int|\nabla u|^2r^{-\alpha-2}e^{\alpha\phi(r)}dx.$$

From Lemma 3 and assuming $a_{ij}^{(2)}(0)=\delta_{ij}$ we may prove the following inequality: there are constants k_4, R_1 and α_0 such that for any $u(x) \in K^{(2)}(R_1)$ and for $\alpha > \alpha_0$

$$\text{(B)} \qquad \begin{aligned} \sum_{|\beta|=2} \int |D^\beta u|^2 r^{-\alpha} e^{\alpha\phi(r)} dx \leqq k_4 \int |L^{(2)}(u)|^2 r^{-\alpha} e^{\alpha\phi(r)} dx \\ + k_4 \alpha^2 \int |\nabla u|^2 r^{-\alpha-2} e^{\alpha\phi(r)} dx. \end{aligned}$$

Using inequalities (A) and (B) it implies the following

Lemma 4. *There are constants k_5, R_1 and α_0 such that for $u \in K^{(4)}(R_1)$ and $\alpha > \alpha_0$*

$$\text{(C)} \qquad \int |L^{(1)} L^{(2)}(u)|^2 r^{-\alpha} e^{\alpha\phi(r)} dx \geqq k_5 m_0^2 \sum_{|\beta|=3} \int |D^\beta u|^2 r^{-\alpha} e^{\alpha\phi(r)} dx.$$

Thus we see the following basic inequality:

Theorem 2. *There are constants k_6, R_1 and α_0 such that for any $u \in K^{(4)}(R_1)$ and for any $\alpha > \alpha_0$*

$$\begin{aligned} \int |L(u)|^2 r^{-\alpha} e^{\alpha\phi(r)} dx \geqq k_6 m_0^2 \Big\{ \sum_{\beta=3} |D^\beta u|^2 r^{-\alpha} e^{\alpha\phi(r)} dx \\ + \sum_{\beta=2} \alpha^2 \int |D^\beta u|^2 r^{-\alpha-2} e^{\alpha\phi(r)} dx + \alpha^4 \sum_{|\beta|=1} \int |D^\beta u|^2 r^{-\alpha-4} e^{\alpha\phi(r)} dx \\ + \alpha^6 \int |u|^2 r^{-\alpha-6} e^{\alpha\phi(r)} dx \Big\}. \end{aligned}$$

Theorem 1 follows directly from Theorem 2 with sufficiently large m_0 such that $k_6 m_0^2 > M$.

Analogous results in more general cases will be proved in a subsequent paper.

References

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